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Quantized GIM Lie algebras and their Lusztig symmetries

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Abstract

We study quantized universal enveloping algebras of Slodowy's GIM Lie algebras. It is shown that they have Lusztig symmetries associated to the corresponding toroidal Weyl groups, and these symmetries satisfy the braid relations given by the toroidal root systems.

1. Introduction

Slodowy's GIM Lie algebra theory is another generalization of simple Lie algebras. The structure matrix of a GIM Lie algebra is a generalized intersection matrix (GIM for short) [9]. The most interesting example of GIM Lie algebras is closely related to the 2-toroidal Lie algebra, the universal central extension of double loop algebra $\mathbb{C}[t_1^{\pm 1}, t_2]^{\pm 1} \otimes \dot{\mathfrak{g}}$, where $\dot{\mathfrak{g}}$ is a complex simple Lie algebra [7]. In this paper we focus on GIMs of simply laced type, which are also called the 2-affinization of a Cartan matrix of ADE type, whose root system has also been studied by K. Saito [10] in a more general situation.

At first we recall how a GIM of 2-affinization type appears naturally from a finite Cartan matrix in terms of root systems (cf. [2]). For basic notation on affine root systems we refer

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to Kac's book [4]. Let $A=(a_{ij})_{i,j=0}^{\ell}$ be an indecomposable generalized Cartan matrix of type $X_\ell^{(1)}$ ($X=\mathrm{A},\mathrm{D},\mathrm{E}$) with $\ell\geqslant 2$. Let $a_0,a_1,\dots,a_\ell$ be the numerical labels of the corresponding Dynkin diagram $S(A)$, while $a_0^\vee,a_1^\vee,\dots,a_\ell^\vee$ be the numerical labels of $S(A^t)$. Then we have the affine Kac–Moody algebra $\mathfrak{g}=\mathfrak{g}(A)$ associated to A with the Chevalley generators e_i, f_i ($0\leqslant i\leqslant\ell$). Let $\mathfrak{h}$ be its Cartan subalgebra, $\Pi=\{\alpha_0,\alpha_1,\dots,\alpha_\ell\}\subset\mathfrak{h}^*$ the set of simple roots, $\Pi^\vee=\{\alpha_0^\vee,\alpha_1^\vee,\dots,\alpha_\ell^\vee\}\subset\mathfrak{h}$ the set of simple coroots, Δ the root system, Q and $Q^\vee$ the root and coroot lattices respectively. The null root is $\delta=\sum_{i=0}^{\ell}a_i\alpha_i\in Q$ and the canonical central element is $K=\sum_{i=0}^{\ell}a_i^\vee\alpha_i^\vee\in Q^\vee$. The scaling element $d\in\mathfrak{h}$ is given by $\alpha_i(d)=\delta_{i0}$ for $i=1,2,\dots,\ell$. Then the normalized invariant bilinear form $(.\mid.)$ on $\mathfrak{g}$ is uniquely determined by the following data: For $i,j=0,1,2,\dots,\ell$, $(\alpha_i^\vee\mid\alpha_j^\vee)=a_{ij}$, $(\alpha_i^\vee\mid d)=\delta_{i0}$, $(d\mid d)=0$. Since $(.\mid.)$ is nondegenerate on $\mathfrak{h}$ it induces an isomorphism $\nu:\mathfrak{h}\to\mathfrak{h}^*$ defined by $\nu(h)(h_1)=(h\mid h_1)$, $h,h_1\in\mathfrak{h}$. Now define an element $\Lambda_0\in\mathfrak{h}^*$ by $\Lambda_0(\alpha_i^\vee)=\delta_{i0}$ for $i=0,1,\dots,\ell$ and $\Lambda_0(d)=0$. By the isomorphism ν we also have the induced bilinear form on $\mathfrak{h}^*$ given by: For $i,j=0,1,2,\dots,\ell$, $(\alpha_i\mid\alpha_j)=a_{ij}$, $(\alpha_i\mid\Lambda_0)=\delta_{i0}$, $(\Lambda_0\mid\Lambda_0)=0$. Denote by $\dot{\mathfrak{h}}$ (respectively $\dot{\mathfrak{h}}^*$) the complex vector space spanned by $\alpha_1^\vee,\dots,\alpha_\ell^\vee$ (respectively $\alpha_1,\dots,\alpha_\ell$). Let $\dot{\mathfrak{g}}$ be the subalgebra of $\mathfrak{g}$ generated by e_i and f_i for $i=1,2,\dots,\ell$. Then $\dot{\mathfrak{g}}$ is the simple finite dimensional Lie algebra whose Cartan matrix is $\dot{A}=(A_{ij})_{i,j=1}^{\ell}$ of type X_ℓ and the corresponding Dynkin diagram $S(\dot{A})$ is obtained from $S(A)$ by deleting the 0th vertex. So $\dot{\mathfrak{h}}=\dot{\mathfrak{g}}\cap\mathfrak{h}$ is the Cartan subalgebra of $\dot{\mathfrak{g}}$, $\dot{\Pi}=\{\alpha_1,\alpha_2,\dots,\alpha_\ell\}$ is the root basis, and $\dot{\Pi}^\vee=\{\alpha_1^\vee,\alpha_2^\vee,\dots,\alpha_\ell^\vee\}$ is the coroot basis for $\dot{\mathfrak{g}}$. The set $\dot{\Delta}=\Delta\cap\dot{\mathfrak{h}}^*$ is the root system and $\dot{Q}=\mathbb{Z}\dot{\Delta}$ is the corresponding root lattice. Furthermore, the restriction of $(.\mid.)_{\mathfrak{h}}$ (respectively $(.\mid.)_{\mathfrak{h}^*}$) to $\dot{\mathfrak{h}}$ (respectively $\dot{\mathfrak{h}}^*$) is positive definite. Also the normalized invariant form on $\mathfrak{g}$ can be restricted to $\dot{\mathfrak{g}}$, and the isomorphism $\nu:\mathfrak{h}\to\mathfrak{h}^*$ is restricted to an isomorphism, again denoted by ν, from $\dot{\mathfrak{h}}$ to $\dot{\mathfrak{h}}^*$. Finally the unique highest root of $\dot{\mathfrak{g}}$ is given by $\theta=\delta-\alpha_0=\sum_{i=0}^{\ell}a_i\alpha_i\in\dot{Q}$. Note that $(\theta\mid\theta)=2$. Assume that $\theta^\vee\in\dot{\mathfrak{h}}$ satisfies that $\nu(\theta^\vee)=\theta$. Then

$$\bigl(\theta^\vee\bigm|\theta^\vee\bigr)=2,\qquad \theta^\vee=K-\alpha_0^\vee=\sum_{i=1}^{l}a_i^\vee\alpha_i^\vee,$$
$$\theta\bigl(\alpha_j^\vee\bigr)=\alpha_j\bigl(\theta^\vee\bigr)=(\alpha_j\mid\theta)=(\alpha_j\mid\delta-\alpha_0)=-a_{j0}=-a_{0j}.$$

Set

$$A=(a_{ij})_{i,j=1}^{\ell+2},\tag{1.1}$$

where

$$a_{\ell+i,j}=a_{j,\ell+i}=-\alpha_j\bigl(\theta^\vee\bigr)\quad\text{for }1\leqslant j\leqslant\ell,\ i=1,2,$$
$$a_{\ell+i,\ell+j}=\theta\bigl(\theta^\vee\bigr)=2\quad\text{for }j=1,2,$$

and the first $\ell\times\ell$ principal minor is exactly $\dot{A}$. Note that $a_{\ell+i,j}=a_{j,\ell+i}\leqslant 0$ for $i=1,2$ and $1\leqslant j\leqslant\ell$ because θ is the highest root. Hence we obtain the 2-affinization of $\dot{A}$ which is an intersection matrix of Slodowy [2,9].

Let H be a complex vector space of dimension $\ell + 4$, H^* be its dual over $\mathbb{C}$, let $\alpha_1, \dots, \alpha_{\ell+2} \in H^*$ and $h_1, \dots, h_{\ell+2} \in H$ be linearly independent collections such that $\alpha_i(h_j) = a_{ij}$, where a_{ij} is given by (1.1). Such a realization exists uniquely due to Kac [4]. Then the GIM Lie algebra $\mathfrak{g}_{\mathrm{gim}}$ of Slodowy is given by the following

Definition 1.1 [9]. The GIM Lie algebra $\mathfrak{g}_{\mathrm{gim}}$ with the structural matrix $A = (a_{ij})_{i,j=1}^{\ell+2}$ given by (1.1) is a Lie algebra with the following presentation.

Generators: $h \in H$, e_i, f_i, $1 \leqslant i \leqslant \ell + 2$.

Relations:

(GIM 1) $[h, h'] = 0$, $h, h' \in H$;
$[h, \mathrm{e}_i] = \alpha_i(h)\mathrm{e}_i$, $1 \leqslant i \leqslant \ell + 2$;
$[h, \mathrm{f}_i] = -\alpha_i(h)\mathrm{e}_i$, $1 \leqslant i \leqslant \ell + 2$;
$[\mathrm{e}_i, \mathrm{f}_i] = h_i$, $1 \leqslant i \leqslant \ell + 2$.

(GIM 2) For $a_{ij} \leqslant 0$ and $1 \leqslant i \neq j \leqslant \ell + 2$, $(\mathrm{ad}\,\mathrm{e}_i)^{1-a_{ij}}\mathrm{e}_j = (\mathrm{ad}\,\mathrm{f}_i)^{1-a_{ij}}\mathrm{f}_j = 0$;
$[\mathrm{e}_i, \mathrm{f}_j] = 0$.

(GIM 3) For $a_{ij} = 2$, i.e., $\ell + 1 \leqslant i \neq j \leqslant \ell + 2$, $(\mathrm{ad}\,\mathrm{e}_i)^3\mathrm{f}_j = (\mathrm{ad}\,\mathrm{f}_i)^3\mathrm{e}_j = 0$;
$[\mathrm{e}_i, \mathrm{e}_j] = [\mathrm{f}_i, \mathrm{f}_j] = 0$.

Note that the corresponding 2-toroidal algebra associated to the simple Lie algebra $\dot{\mathfrak{g}}$ is an epimorphism image of $\mathfrak{g}_{\mathrm{gim}}$ (cf. [7]).

Now we review the toroidal Weyl group of $\mathfrak{g}_{\mathrm{gim}}$ based on the work [8]. Keep notation as above. Define $\Lambda_1, \Lambda_2 \in H^*$ by $\Lambda_i(h_j) = 0$ for $i = 1, 2$, $1 \leqslant j \leqslant \ell$ and $\Lambda_i(h_{\ell+j}) = \delta_{ij}$ for $i, j = 1, 2$. It follows that $\alpha_1, \dots, \alpha_{\ell+2}, \Lambda_1, \Lambda_2$ form a basis of H^*. Similarly, define $d_1, d_2 \in H$ by $\alpha_i(d_j) = 0$ for $j = 1, 2$, $1 \leqslant i \leqslant \ell$ and $\alpha_{\ell+i}(d_j) = \delta_{ij}$ for $i, j = 1, 2$. Then $h_1, \dots, h_{\ell+2}, d_i, d_2$ form a basis of H. Introduce $\delta_1, \delta_2 \in H^*$ such that $\alpha_{\ell+i} = \delta_i - \theta$ for $i = 1, 2$. Then it is easy to see that $\delta_i(d_j) = \delta_{ij}$ since $\theta(d_j) = 0$. Also introduce $c_1, c_2 \in H$ such that $h_{\ell+i} = c_i - \theta^\vee$ for $i = 1, 2$. It is easy to see that $\Lambda_i(c_j) = \delta_{ij}$ since $\Lambda_i(\theta^\vee) = 0$. Hence there is a symmetric bilinear nondegenerate form $(.\mid.)$ on H^* such that, for $1 \leqslant i, j \leqslant \ell$ and $k, k' = 1, 2$,

$$(\alpha_i \mid \alpha_j) = a_{ij}, \quad (\alpha_i \mid \delta_k) = (\alpha_i \mid \Lambda_k) = (\delta_k \mid \delta_{k'}) = (\Lambda_k \mid \Lambda_{k'}) = 0, \quad (\delta_k \mid \Lambda_{k'}) = \delta_{kk'}.$$

Note that $(\delta_i \mid \alpha_j) = 0$ for $1 \leqslant j \leqslant \ell + 2$ and $(\Lambda_k \mid \alpha_{\ell+j}) = \delta_{kj}$ for $k, j = 1, 2$. Since $(.\mid.)$ is nondegenerate it determines an isomorphism $\nu^{-1}: H^* \to H$ given by $\beta(\nu^{-1}(\alpha)) = (\alpha \mid \beta)$ for $\alpha, \beta \in H^*$. It is easy to see that $\nu^{-1}(\alpha_i) = h_i$ for $1 \leqslant i \leqslant \ell + 2$ and $\nu^{-1}(\Lambda_i) = d_i$, $\nu^{-1}(\delta_i) = c_i$ for $i = 1, 2$. Set

$$\Gamma = \bigoplus_{i=1}^{\ell+2} \mathbb{Z}\alpha_i = \bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i \oplus \mathbb{Z}\delta_1 \oplus \mathbb{Z}\delta_2 \subset H^*,$$
$$Q^{[2]} = \dot{Q} \oplus \mathbb{Z}\delta_1 \oplus \mathbb{Z}\delta_2, \qquad \triangle^{[2]} = \triangle = \dot{\triangle} \oplus \mathbb{Z}\delta_1 \oplus \mathbb{Z}\delta_2. \tag{1.2}$$

Then $Q^{[2]}$ is the 2-toroidal root lattice and elements of $\triangle$ are called real roots of $\mathfrak{g}_{\mathrm{gim}}$. For any $\mu \in \triangle$, let r_μ be the fundamental reflection in μ with respect to $(. \mid .)$, i.e., $r_\mu(\lambda) = \lambda - (\lambda \mid \mu)\mu$ for any $\lambda \in \Gamma$. Then the toroidal Weyl group of $\mathfrak{g}_{\mathrm{gim}}$ is $W = \langle r_i \mid 1 \leqslant i \leqslant \ell + 2\rangle = \langle r_\mu \mid \mu \in \triangle\rangle$, where $r_i = r_{\alpha_i}$ for $1 \leqslant i \leqslant \ell + 2$. In [8] it is has been shown that these reflections satisfy Coxter group relations and W is a quotient of an indefinite Weyl group with a highly nontrivial kernel.

The motivation to study the q-analogue of $\mathfrak{g}_{\mathrm{gim}}$ comes from the construction of Ginzburg–Kapranov–Vasserot in [3]. They defined firstly the q-analogue, i.e., quantum toroidal algebras of 2-toroidal Lie algebras by using a presentation of Drinfel'd type with infinitely many generators and defining relations. There have been important results about quantum toroidal algebras (cf., for example, [11]). Compared with the extensive study of quantized affine Kac–Moody algebras, the theory of quantum toroidal algebras is far from well developed, partly because 2-toroidal Lie algebras has no finite presentation available at present and no natural triangular decomposition in a fashion of Kac–Moody algebras. Since a 2-toroidal Lie algebra is a quotient of the corresponding GIM Lie algebra, one may expect that the q-analogue of GIM Lie algebra should provide some ideas to study quantum toroidal algebras.

Although quantized GIM Lie algebras (see Definition 2.1 below) has a finite presentation like usual quantized affine algebras, there are essential differences between quantized GIM Lie algebras and quantized affine algebras. For example, quantized GIM Lie algebras are not Hopf algebras with respect to the usual definition of comultiplication (see Remark 2.1 below). Also they have no natural triangular decomposition. Thus many results such as integrable representation theory of quantized affine algebras need to be essentially modified to quantized GIM Lie algebras. However, thanks to results of Moody–Shi on toroidal Weyl groups [8], it is possible to establish Lusztig symmetries and their braid relations of quantized GIM Lie algebras. Indeed, due to Lusztig's construction, for quantized affine algebras, their Lusztig symmetries are closely related to Weyl groups. There is no need to mention that Lusztig symmetries are very important to quantum group theory. It is not difficult to write down explicitly these Lusztig symmetries for quantized GIM Lie algebras (see Theorem 2.1 below), but Lusztig's proof, which uses Verma module theory and depends heavily on triangular decomposition, seems not to be applicable directly to GIM cases. So we adopt a more combinatorial way, which involves a large quantity of computations. Indeed, most of our computation is handling the Serre relations. When restricted to affine case, we obtain another direct proof of Lusztig's results on symmetries of quantized affine algebras (of simply laced type), without using representation theory.

We would like to point out that K. Miki has studied the braid group action on quantum toroidal algebras of $A_l^{(1)}$ type [5], which knits Lusztig symmetries of the horizontal and vertical parts of the quantum toroidal algebra by using Beck's technique [1]. In the case of quantized GIM Lie algebras, Beck's construction would be applicable to obtain a loop-like presentation of quantized GIM Lie algebras and hence action of Lusztig symmetries on these loop-like generators can be written in a more explicit way.

The paper is organized as follows. In Section 2 we give out the definition of quantized GIM Lie algebras and the main results on Lusztig symmetries. In Sections 3–6 we prove Theorem 2.1 and in Section 7 we prove Theorem 2.2, i.e., Lusztig symmetries satisfy braid relations. Since, the toroidal Weyl group of a quantized GIM Lie algebra is a highly nontrivial quotient of an indefinite Weyl group, the Lusztig symmetries may satisfy other relations besides braid relations. It would be interesting to find out those relations.

2. Definitions and the main results

Let q be an indeterminant. Then we have the usual notation $[n]=(q^n-q^{-n})/(q-q^{-1})$ and $[n]!=[n][n-1]\cdots[2][1]$. In particular, we have that

$$[2]=q+q^{-1},\qquad [3]=q^2+q^{-2}+1,\qquad [2]^2=[3]+1.$$

Also, for any letter x, denote $x^{(n)}=x^n/[n]!$. Now we give out the definition of the q-analogue of $\mathfrak{g}_{\mathrm{gim}}$ associated to the GIM matrix $A=(a_{ij})_{i,j=1}^{\ell+2}$ given by (1.1) and the lattice Γ given by (1.2) as following.

Definition 2.1. The algebra $\mathrm{U}_q=\mathrm{U}_q(\mathfrak{g}_{\mathrm{gim}})$ is a unitary associative algebra over $\mathbb{Q}(q)$ with the following presentation.

Generators: K_α $(\alpha\in\Gamma)$, E_j, F_j $(1\leqslant j\leqslant \ell+2)$, $C_i^{\pm 1/2}$, $D_i^{\pm 1}$ $(i=1,2)$.

Relations:

(Re. 1) $[K_\alpha, D_i^{\pm 1}]=0$, $K_\alpha K_\beta=K_{\alpha+\beta}$, $K_0=1=D_iD_i^{-1}=D_i^{-1}D_i$,
$C_i^{\pm 1/2}$ is central and $(C_i^{\pm 1/2})^2=K_{\delta_i}^{\pm 1}$, $i=1,2$;
$K_\alpha E_j K_\alpha^{-1}=q^{\alpha(h_j)}E_j$, $K_\alpha F_j K_\alpha^{-1}=q^{-\alpha(h_j)}F_j$, $1\leqslant j\leqslant \ell+2$;
$D_iE_jD_i^{-1}=q^{\delta_{\ell+i,j}}E_j$, $D_iF_jD_i^{-1}=q^{-\delta_{\ell+i,j}}F_j$, $i=1,2$, $1\leqslant j\leqslant \ell+2$;
$[E_j,F_j]=\frac{K_j-K_j^{-1}}{q-q^{-1}}$, where $K_j=K_{\alpha_j}$, $1\leqslant j\leqslant \ell+2$.

(Re. 2) For $a_{ij}\leqslant 0$ and $1\leqslant i\neq j\leqslant \ell+2$, $[E_i,F_j]=0$,

$$\sum_{s=0}^{1-a_{ij}}(-1)^s E_i^{(1-a_{ij}-s)}E_jE_i^{(s)}=0=\sum_{s=0}^{1-a_{ij}}(-1)^s F_i^{(1-a_{ij}-s)}F_jF_i^{(s)}. \tag{2.1}$$

(Re. 3) For $a_{ij}=2$, $i\neq j$, i.e., $\ell+1\leqslant i\neq j\leqslant \ell+2$,

$$\sum_{s=0}^{3}(-1)^s E_i^{(3-s)}F_jE_i^{(s)}=0=\sum_{s=0}^{3}(-1)^s F_i^{(3-s)}E_jF_i^{(s)}, \tag{2.2}$$

$$[E_i,E_j]=[F_i,F_j]=0. \tag{2.3}$$

The relations (2.1)–(2.3) are called Serre relations. It follows that there is a $\mathbb{C}$-algebra automorphism Φ of U_q given by:

$$\begin{gathered}\Phi(E_j)=F_j \quad (1\leqslant j\leqslant \ell+2), \qquad \Phi(F_j)=E_j \quad (1\leqslant j\leqslant \ell+2),\\ \Phi(K_\alpha)=K_\alpha \quad (\alpha\in\Gamma),\\ \Phi\big(C_i^{\pm 1/2}\big)=C_i^{\pm 1/2}, \quad \Phi\big(D_i^{\pm 1}\big)=D_i^{\pm 1} \quad (i=1,2), \qquad \Phi(q)=q^{-1}. \end{gathered}\tag{2.4}$$

Also there is a $\mathbb{C}$-algebra anti-automorphism Ω of U_q given by:

$$\begin{gathered}\Omega(E_j)=F_j \quad (1\leqslant j\leqslant \ell+2), \qquad \Omega(F_j)=E_j \quad (1\leqslant j\leqslant \ell+2),\\ \Omega(K_\alpha)=K_{-\alpha} \quad (\alpha\in\Gamma),\\ \Omega\big(C_i^{\mp 1/2}\big)=C_i^{\pm 1/2}, \quad \Omega\big(D_i^{\pm 1}\big)=D_i^{\mp 1} \quad (i=1,2), \qquad \Omega(q)=q^{-1}. \end{gathered}\tag{2.5}$$

Notation. From now on we denote $K_{\sum t_i\alpha_i}$ by $K_{\sum t_i i}$ for any $\sum t_i\alpha_i\in\Gamma$.

Let $\mathrm{U}_q^{(i)}$ $(i=1,2)$ be the subalgebra of U_q generated by K_α $(\alpha\in\Gamma)$, E_k, F_k $(k=1,2,\dots,\ell,\ell+i)$, $C_i^{\pm 1/2}$, $D_i^{\pm 1}$. Then $\mathrm{U}_q^{(i)}\simeq \mathrm{U}_q(\mathfrak{g})$, the quantized enveloping algebra of the affine Lie algebra $\mathfrak{g}$. This fact may be shown by the following theorem and we don't use it at this moment. The first result of this paper may be formulated as follows.

Theorem 2.1. *For each* $1\leqslant i\leqslant \ell+2$, *there is an automorphism* T_i *of* U_q *given by*:

$$T_i(E_i)=-F_iK_i,$$

$$T_i(E_j)=\begin{cases}\sum_{s=0}^{-a_{ij}}(-1)^{s-a_{ij}}q^{-s}E_i^{(-a_{ij}-s)}E_jE_i^{(s)} & \text{for } a_{ij}\leqslant 0,\\ \sum_{s=0}^{2}(-1)^{2-s}q^{2-s}F_i^{(2-s)}E_jF_i^{(s)} & \text{for } a_{ij}=2 \text{ and } i\neq j,\end{cases}\tag{2.6}$$

$$T_i(F_i)=-K_i^{-1}E_i,$$

$$T_i(F_j)=\begin{cases}\sum_{s=0}^{-a_{ij}}(-1)^{s-a_{ij}}q^{s}F_i^{(s)}F_jF_i^{(-a_{ij}-s)} & \text{for } a_{ij}\leqslant 0,\\ \sum_{s=0}^{2}(-1)^{2-s}q^{s-2}E_i^{(s)}F_jE_i^{(2-s)} & \text{for } a_{ij}=2 \text{ and } i\neq j,\end{cases}\tag{2.7}$$

$$T_i(K_\alpha)=K_{r_i(\alpha)} \quad (\alpha\in\Gamma),$$

$$T_i\big(C_j^{\pm 1/2}\big)=C_j^{\pm 1/2}, \quad T_i\big(D_j^{\pm 1}\big)=D_j^{\pm 1}K_i^{\mp\delta_{i,\ell+j}} \quad (j=1,2).$$

Here r_i *is the fundamental reflection on* Γ. *The inverse* T_i' *of* T_i *is given by*:

$$T_i'(E_i)=-K_i^{-1}F_i,$$

$$T_i'(E_j)=\begin{cases}\sum_{s=0}^{-a_{ij}}(-1)^{s-a_{ij}}q^{-s}E_i^{(s)}E_jE_i^{(-a_{ij}-s)} & \text{for } a_{ij}\leqslant 0,\\ \sum_{s=0}^{2}(-1)^{2-s}q^{2-s}F_i^{(s)}E_jF_i^{(2-s)} & \text{for } a_{ij}=2 \text{ and } i\neq j,\end{cases}\tag{2.8}$$

$$T_i'(F_i)=-E_iK_i,$$

$$T_i'(F_j)=\begin{cases}\sum_{s=0}^{-a_{ij}}(-1)^{s-a_{ij}}q^{s}F_i^{(-a_{ij}-s)}F_jF_i^{-s)} & \text{for } a_{ij}\leqslant 0,\\ \sum_{s=0}^{2}(-1)^{2-s}q^{s-2}E_i^{(2-s)}F_jE_i^{(s)} & \text{for } a_{ij}=2 \text{ and } i\neq j,\end{cases}\tag{2.9}$$

$$T_i'(K_\alpha) = K_{r_i(\alpha)} \quad (\alpha \in \Gamma),$$
$$T_i'\big(C_j^{\pm 1/2}\big) = C_j^{\pm 1/2}, \quad T_i'\big(D_j^{\pm 1}\big) = D_j^{\pm 1} K_i^{\mp \delta_{i,\ell+j}} \quad (j = 1, 2).$$

These automorphisms are called Lusztig symmetries of U_q. The proof will be given in following sections. The outline of proof is that, at first we shall show that T_i is an algebra endomorphism of U_q by checking that it respects defining relations (Re. 1)–(Re. 3) of U_q (T_i' is similarly treated), then we show that $T_i T_i' = 1 = T_i' T_i$.

Theorem 2.2. *The Lusztig symmetries T_i $(1 \leqslant i \leqslant \ell + 2)$ satisfy the braid group relation, i.e.,*

$$T_i T_j = T_j T_i \quad \textit{for } a_{ij} = 0,$$
$$T_i T_j T_i = T_j T_i T_i \quad \textit{for } a_{ij} = -1.$$

Remark 2.1. Unlike the usual quantized enveloping algebras of Kac–Moody algebras, U_q is not a Hopf algebra with the usual definition given by

$$\triangle(E_i) = E_i \otimes 1 + K_i \otimes E_i, \qquad \triangle(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \qquad \triangle(K_\alpha) = K_\alpha \otimes K_\alpha.$$

Then $\triangle$ is not an algebra homomorphism from U_q to $\mathrm{U}_q \otimes \mathrm{U}_q$ since it does not respect the Serre relations (2.2) and (2.3). Indeed, if $a_{ij} = 2$ and $i \neq j$ then

$$\begin{aligned}
&\big(\triangle(E_i)\big)^3 \triangle(F_j) - [3]\big(\triangle(E_i)\big)^2 \triangle(F_j)\triangle(E_i) \\
&\quad + [3]\triangle(E_i)\triangle(F_j)\big(\triangle(E_i)\big)^2 - \triangle(F_j)\big(\triangle(E_i)\big)^3 \\
&= [3]\big(q^{-2} - 1\big)\big\{K_{2i} E_i \otimes \big[E_i^2 F_j - \big(1 + q^{-2}\big)E_i F_j E_i + q^{-2} F_j E_i^2\big] \\
&\quad - K_i\big[E_i^2 F_j - \big(1 + q^2\big)E_i F_j E_i + q^2 F_j E_i^2\big] \otimes K_j^{-1} E_i\big\} \neq 0, \quad \text{and} \\
&\quad \triangle(E_i)\triangle(E_j) - \triangle(E_j)\triangle(E_i) = \big(q^{-2} - 1\big)\{K_j E_i \otimes E_j - K_i E_j \otimes E_i\} \neq 0.
\end{aligned}$$

The case of $\triangle(F_i)$, $\triangle(E_j)$ is similar.

3. The first type of defining relations

Keep notation as in last section. In this section we shall show that, for each $1 \leqslant j \leqslant \ell + 2$, T_j respects the defining relation (Re. 1). By the definition of fundamental reflections of the Weyl group, it is easy to see that T_j respects the defining relations: $[K_\alpha, K_\beta] = [K_\alpha, D_i] = 0$, $K_\alpha K_\beta = K_{\alpha+\beta}$, $K_0 = 1$ and $T_j(C_i^{\pm 1/2})$ is still central in U_q. At first we prove the following

Lemma 3.1. *For any* $1 \leqslant i, j \leqslant \ell + 2$ *and* $\alpha \in \Gamma$, *it holds that*

$$T_j(K_\alpha)T_j(E_i)T_j\left(K_\alpha^{-1}\right) = q^{\alpha(h_i)}T_j(E_i), \tag{3.1}$$

$$T_j(K_\alpha)T_j(F_i)T_j\left(K_\alpha^{-1}\right) = q^{-\alpha(h_i)}T_j(F_i). \tag{3.2}$$

Proof. We prove (3.1) while (3.2) is obtained from (3.1) by using the involution Φ of U_q. The case that $a_{ij} \leqslant 0$ is known and can be checked directly. So we assume that $a_{ij} = 2$ and $i \neq j$. Note that

$$E_i K_{r_j(\alpha)}^{-1} = q^{\alpha(h_i)-2\alpha(h_j)} K_{r_j(\alpha)}^{-1} E_i, \qquad F_j^{(s)} K_{r_j(\alpha)}^{-1} = q^{s\alpha(h_j)} K_{r_j(\alpha)}^{-1} E_i.$$

By definition (2.6) of T_j it follows that it holds that

$$\begin{aligned}
&T_j(K_\alpha)T_j(E_i)T_j\left(K_\alpha^{-1}\right)\\
&\quad = K_{r_j(\alpha)} \sum_{s=0}^{2} (-1)^{2-s} q^{2-s} F_j^{(2-s)} E_i F_j^{(s)} K_{r_j(\alpha)}^{-1}\\
&\quad = K_{r_j(\alpha)} \sum_{s=0}^{2} (-1)^{s-a_{ij}} q^{2-s} q^{s\alpha(h_j)+\alpha(h_i)-2\alpha(h_j)+(2-s)\alpha(h_j)} K_{r_j(\alpha)}^{-1} F_j^{(2-s)} E_i F_j^{(s)}\\
&\quad = q^{\alpha(h_i)} T_j(E_i).
\end{aligned}$$

This completes the proof. $\square$

Lemma 3.2. *For any* $1 \leqslant i, j \leqslant \ell + 2$ *and* $k = 1, 2$ *it holds that*

$$T_j(D_k)T_j(E_i)T_j\left(D_k^{-1}\right) = q^{\delta_{\ell+k,i}}T_j(E_i), \tag{3.3}$$

$$T_j(D_k)T_j(F_i)T_j\left(D_k^{-1}\right) = q^{-\delta_{\ell+k,i}}T_j(F_i). \tag{3.4}$$

Proof. We show that $T_j(D_1)T_j(E_i)T_j(D_1^{-1}) = q^{\delta_{\ell+1,i}}T_j(E_i)$. The remaining formulae are similar. This is standard. For example, assume that $i \neq j$ and $a_{ij} \leqslant 0$. Then

$$\begin{aligned}
&T_j(D_1)T_j(E_i)T_j\left(D_1^{-1}\right)\\
&\quad = D_1 K_j^{-\delta_{\ell+1,j}} \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q^{-s} E_j^{(-a_{ij}-s)} E_i E_j^{(s)} D_1^{-1} K_j^{\delta_{\ell+1,j}}\\
&\quad = D_1 K_j^{-\delta_{\ell+1,j}} \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q^{-s} q^{s\delta_{\ell+1,j}+\delta_{\ell+1,i}-(s+a_{ij})\delta_{\ell+1,j}}\\
&\qquad \times D_1^{-1} E_j^{(-a_{ij}-s)} E_i E_j^{(s)} D_1 K_j^{\delta_{\ell+1,j}}
\end{aligned}$$

$$= q^{\delta_{\ell+1,i}-a_{ij}\delta_{\ell+1,j}} K_j^{-\delta_{\ell+1,j}} \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q^{-s}$$

$$\times q^{(-2s+2(s+a_{ij})-a_{ij})\delta_{\ell+1,j}} K_j^{\delta_{\ell+1,j}} E_j^{(-a_{ij}-s)} E_i E_j^{(s)}$$

$$= q^{\delta_{\ell+1,i}} T_j(E_i).$$

The case that $a_{ij} = 2$, $i \neq j$, is similar. □

Now we proceed to show that T_j respects the remaining relation in (Re. 1): $[E_i, F_i] = (K_i - K_i^{-1})/(q - q^{-1})$, i.e., we shall prove the following

Proposition 3.1. *For any $1 \leqslant i, j \leqslant \ell + 2$, it holds that*

$$\bigl[T_j(E_i), T_j(F_i)\bigr] = \frac{K_{r_j(\alpha_i)} - K_{r_j(\alpha_i)}^{-1}}{q - q^{-1}}. \tag{3.5}$$

At first, we need the following facts (the special cases of formulae in Lusztig's book [6, Corollary 3.1.9]), and they will be used frequently in later sections. Also we shall fix some notation for further computations.

For any $1 \leqslant j \leqslant \ell + 2$, set

$$Q_j = \frac{K_j - K_j^{-1}}{q - q^{-1}}, \tag{3.6}$$

$$x_j = \frac{(1+q^{-2})K_j - (1+q^2)K_j^{-1}}{q^2 - q^{-2}}, \tag{3.7}$$

$$\bar{x}_j = \frac{(1+q^{2})K_j - (1+q^{-2})K_j^{-1}}{q^2 - q^{-2}}, \tag{3.8}$$

$$P_j = \frac{(1+q^{-2})K_j^2 + (1+q^2)K_j^{-2}}{(q^2 - q^{-2})^2} - \frac{1}{(q - q^{-1})^2}. \tag{3.9}$$

Lemma 3.3. *Keep notation as above. For any $1 \leqslant j \leqslant \ell + 2$, it holds that*

$$E_j F_j^{(2)} = F_j^{(2)} E_j + F_j x_j,$$

$$F_j E_j^{(2)} = E_j^{(2)} F_j + E_j \bar{x}_j,$$

$$E_j^{(2)} F_j^{(2)} = F_j^{(2)} E_j^{(2)} + F_j E_j Q_j + P_j.$$

Proof. It follows by a direct computation by $[E_j, F_j] = (K_j - K_j^{-1})/(q - q^{-1})$. □

If either $1 \leqslant i = j \leqslant \ell + 2$ or $a_{ij} \leqslant 0$ then (3.5) follows directly by definitions (2.6), (2.7) of T_i. So it suffices to consider the remaining case: $a_{ij} = 2$ but $i \neq j$. So, Proposition 3.1 is reduced to the following

Lemma 3.4. *Assume that $a_{ij} = 2$ but $i \neq j$. Then*

$$\bigl[T_j(E_i), T_j(F_i)\bigr] = \frac{K_{r_j(\alpha_i)} - K_{r_j(\alpha_i)}^{-1}}{q - q^{-1}} = \frac{K_{i-2j} - K_{i-2j}^{-1}}{q - q^{-1}}, \tag{3.10}$$

where $K_{i-2j} = K_{\alpha_i - 2\alpha_j}$.

Proof. Note that in this case we have that $E_i E_j = E_j E_i$ and $F_i F_j = F_j F_i$. By definitions (2.6), (2.7) of T_j we have that $[T_j(E_i), T_j(F_i)] = \sum_{t=1}^{9} Y_t$, where

$$\begin{aligned}
Y_1 &= F_j^{(2)} E_i F_i E_j^{(2)} - F_i E_j^{(2)} F_j^{(2)} E_i,\\
Y_2 &= -q\bigl[F_j^{(2)} E_i E_j F_i E_j - E_j F_i E_j F_j^{(2)} E_i\bigr],\\
Y_3 &= -q\bigl[F_j E_i F_j E_j^{(2)} F_i - E_j^{(2)} F_i F_j E_i F_j\bigr],\\
Y_4 &= -q^{-1}\bigl[F_j E_i F_j F_i E_j^{(2)} - F_i E_j^{(2)} F_j E_i F_j\bigr],\\
Y_5 &= -q^{-1}\bigl[E_i F_j^{(2)} E_j F_i E_j - E_j F_i E_j E_i F_j^{(2)}\bigr],\\
Y_6 &= q^2\bigl[F_j^{(2)} E_i E_j^{(2)} F_i - E_j^{(2)} F_i F_j^{(2)} E_i\bigr],\\
Y_7 &= q^{-2}\bigl[E_i F_j^{(2)} F_i E_j^{(2)} - F_i E_j^{(2)} E_i F_j^{(2)}\bigr],\\
Y_8 &= F_j E_i F_j E_j F_i E_j - E_j F_i E_j F_j E_i F_j,\\
Y_9 &= E_i F_j^{(2)} E_j^{(2)} F_i - E_j^{(2)} F_i E_i F_j^{(2)}.
\end{aligned}$$

We compute Y_t $(1 \leqslant t \leqslant 9)$ by using the facts $E_i E_j = E_j E_i$, $F_i F_j = F_j F_i$ and Lemma 3.3 frequently, to express these terms as combinations of some PBW-basis-like elements.

$$\begin{aligned}
Y_1 &= F_j^{(2)}(F_i E_i + Q_i) E_j^{(2)} - F_i\bigl(F_j^{(2)} E_j^{(2)} + F_j E_j Q_j + P_j\bigr) E_i\\
&= F_j^{(2)} E_j^{(2)} \frac{q^4 K_i - q^{-4} K_i^{-1}}{q - q^{-1}} - F_i F_j E_j Q_j E_i - F_i P_j E_i,
\end{aligned} \tag{3.11}$$

where Q_j is given by (3.6), P_j is given by (3.9).

$$\begin{aligned}
Y_2 &= -q\bigl[F_j^{(2)} E_i E_j F_i E_j - E_j F_i\bigl(F_j^{(2)} E_j + F_j x_j\bigr) E_i\bigr]\\
&= -q\bigl[F_j^{(2)} E_j (F_i E_i + Q_i) E_j - E_j F_j^{(2)} F_i E_i E_j - E_j F_i F_j x_j E_i\bigr]\\
&= -q\bigl[\bigl(F_j^{(2)} E_j - E_j F_j^{(2)}\bigr) F_i E_i E_j + F_j^{(2)} E_j Q_i E_j - E_j F_i F_j x_j E_i\bigr]
\end{aligned}$$

$$= -q\Bigg[\big(q+q^{-1}\big)F_j^{(2)}E_j^{(2)}\frac{q^2K_i - q^{-2}K_i^{-1}}{q-q^{-1}} \\ - F_jF_iE_iE_j\bar{x}_j - E_jF_jF_iE_i\bar{x}_j\Bigg], \tag{3.12}$$

where $\bar{x}_j$ is given by (3.8). Similarly we have that

$$Y_3 = -q\Bigg[\big(q+q^{-1}\big)F_j^{(2)}E_j^{(2)}\frac{q^{-2}K_i - q^2K_i^{-1}}{q-q^{-1}} \\ + \big(q+q^{-1}\big)(F_jE_jQ_j + P_j)\frac{q^{-2}K_i - q^2K_i^{-1}}{q-q^{-1}} - E_jE_iF_iF_jx_j \\ - (E_jF_jF_iE_i + E_jF_jQ_i - Q_jF_iE_i - Q_jQ_i)x_j\Bigg], \tag{3.13}$$

$$Y_4 = -q^{-1}\Bigg[\big(q+q^{-1}\big)F_j^{(2)}E_j^{(2)}\frac{q^2K_i - q^{-2}K_i^{-1}}{q-q^{-1}} - F_jF_iE_iE_j\bar{x}_j \\ - (E_iF_iF_jE_j + E_iF_iQ_j - Q_iF_jE_j - Q_iQ_j)\bar{x}_j\Bigg]. \tag{3.14}$$

$$Y_5 = -q^{-1}\Bigg[\big(q+q^{-1}\big)F_j^{(2)}E_j^{(2)}\frac{q^{-2}K_i - q^2K_i^{-1}}{q-q^{-1}} \\ + \big(q+q^{-1}\big)(F_jE_jQ_j + P_j)\frac{q^{-2}K_i - q^2K_i^{-1}}{q-q^{-1}} - E_jE_iF_iF_jx_j \\ - E_iF_iF_jE_jx_j\Bigg], \tag{3.15}$$

$$Y_6 = q^2\big[F_j^{(2)}E_j^{(2)}Q_i - E_jF_jF_iE_iQ_j + Q_jF_iE_iQ_j - P_jF_iE_i\big], \tag{3.16}$$

$$Y_7 = q^{-2}\big[F_j^{(2)}E_j^{(2)}Q_i - E_iF_iF_jE_jQ_j + Q_iF_jE_jQ_j - F_iE_iP_j\big], \tag{3.17}$$

$$Y_8 = \big(q+q^{-1}\big)^2F_j^{(2)}E_j^{(2)}Q_i - E_jF_jF_iE_iQ_j - E_iF_iF_jE_jQ_j \\ - F_jF_iE_iE_jQ_j - E_jE_iF_iF_jQ_j \\ + F_jQ_jE_jQ_i + Q_jF_jE_jQ_i - F_jQ_iE_jQ_j + E_jQ_iF_jQ_j, \tag{3.18}$$

$$Y_9 = F_j^{(2)}E_j^{(2)}\frac{q^{-4}K_i - q^4K_i^{-1}}{q-q^{-1}} + (F_jE_jQ_j + P_j)\frac{q^{-4}K_i - q^4K_i^{-1}}{q-q^{-1}} \\ - E_iE_jF_jQ_jF_i + E_iQ_j^2F_i - E_iP_jF_i. \tag{3.19}$$

The remaining computations devote to showing that there are no nonzero summands in $[T_j(E_i), T_j(F_i)]$ other than $(K_{i-2j} - K_{i-2j}^{-1})/(q-q^{-1})$.

(1) There are no summands containing $F_j^{(2)}E_j^{(2)}$.

Indeed, by (3.11) and (3.19), the right coefficient of $F_j^{(2)}E_j^{(2)}$ in Y_1+Y_9 is

$$\frac{q^4K_i-q^{-4}K_i^{-1}}{q-q^{-1}}+\frac{q^{-4}K_i-q^4K_i^{-1}}{q-q^{-1}}=\left(q^4+q^{-4}\right)Q_i;$$

by (3.12) and (3.13), the right coefficient of $F_j^{(2)}E_j^{(2)}$ in Y_2+Y_3 is

$$-q\left(q+q^{-1}\right)\left[\frac{q^2K_i-q^{-2}K_i^{-1}}{q-q^{-1}}+\frac{q^{-2}K_i-q^2K_i^{-1}}{q-q^{-1}}\right]=-q\left(q+q^{-1}\right)\left(q^2+q^{-2}\right)Q_i;$$

by (3.14) and (3.15), the right coefficient of $F_j^{(2)}E_j^{(2)}$ in Y_4+Y_5 is

$$\begin{aligned}&-q^{-1}\left(q+q^{-1}\right)\left[\frac{q^2K_i-q^{-2}K_i^{-1}}{q-q^{-1}}+\frac{q^{-2}K_i-q^2K_i^{-1}}{q-q^{-1}}\right]\\&\quad=-q^{-1}\left(q+q^{-1}\right)\left(q^2+q^{-2}\right)Q_i;\end{aligned}$$

by (3.16) and (3.17), the right coefficient of $F_j^{(2)}E_j^{(2)}$ in Y_6+Y_7 is $(q^2+q^{-2})Q_i$; and by (3.18), the right coefficient of $F_j^{(2)}E_j^{(2)}$ in Y_8 is $(q+q^{-1})^2Q_i$. So, the right coefficient of $F_j^{(2)}E_j^{(2)}$ in $[T_j(E_i),T_j(F_i)]$ is

$$\left(q^4+q^{-4}-\left(q+q^{-1}\right)^2\left(q^2+q^{-2}\right)+q^2+q^{-2}+\left(q+q^{-1}\right)^2\right)Q_i=0.$$

(2) By (3.11), (3.12), (3.14) and (3.18), the right coefficient of $F_iF_jE_iE_j$ in $[T_j(E_i),T_j(F_i)]$ is

$$-\frac{q^2K_j-q^{-2}K_j^{-1}}{q-q^{-1}}+q\bar{x}_j+q^{-1}\bar{x}_j-Q_j=0.$$

(3) By (3.19), (3.13), (3.15) and (3.18), the right coefficient of $E_iE_jF_iF_j$ in $[T_j(E_i),T_j(F_i)]$ is

$$-\frac{q^{-2}K_j-q^2K_j^{-1}}{q-q^{-1}}+qx_j+q^{-1}x_j-Q_j=0.$$

(4) By (3.12), (3.13), (3.16) and (3.18), the right coefficient of $E_jF_jF_iE_i$ in $[T_j(E_i),T_j(F_i)]$ is

$$q\bar{x}_j+qx_j-q^2Q_j-Q_j=\left(q^2+1\right)Q_j-\left(q^2+1\right)Q_j=0.$$

(5) By (3.14), (3.15), (3.17) and (3.18), the right coefficient of $E_iF_iF_jE_j$ in $[T_j(E_i),T_j(F_i)]$ is

$$q^{-1}\bar{x}_j+q^{-1}x_j-q^{-2}Q_j-Q_j=\left(q^{-2}+1\right)Q_j-\left(q^{-2}+1\right)Q_j=0.$$

So, by facts (1)–(5) as above and (3.11)–(3.19) it follows that

$$\big[T_j(E_i), T_j(F_i)\big] = F_iE_iA_1 + E_iF_iA_2 + F_jE_jB_1 + E_jF_jB_2 + C, \tag{3.20}$$

where A_1, A_2, B_1, B_2, C contain no letters E_i, F_i, E_j, F_j.

(i) Consider the coefficient of F_iE_i in $[T_j(E_i), T_j(F_i)]$.

There is a term in Y_1 (cf. (3.11)):

$$-F_iP_jE_i = -\frac{1}{[2]^2}F_iE_i\frac{(q^2+q^4)K_{2j} + (q^{-4}+q^{-2})K_{-2j}}{(q-q^{-1})^2} + \frac{1}{(q-q^{-1})^2}F_iE_i.$$

There is a term in Y_3 (cf. (3.13)):

$$\begin{aligned} -qQ_jF_iE_ix_j &= F_iE_i(-qQ_jx_j) \\ &= -\frac{1}{[2]^2}F_iE_i\frac{(1+q^2)(q^{-2}+1)K_{2j} + (1+q^2)^2K_{-2j}}{(q-q^{-1})^2} \\ &\quad + \frac{1+q^2}{(q-q^{-1})^2}F_iE_i. \end{aligned}$$

There is a summand in Y_6 (cf. (3.16)):

$$\begin{aligned} &-q^2P_jF_iE_i + q^2Q_jF_iE_iQ_j \\ &\quad = -\frac{1}{[2]^2}F_iE_i\frac{[(1+q^2)-(1+q^2)^2]K_{2j} + [(q^2+q^4)-(1+q^2)^2]K_{-2j}}{(q-q^{-1})^2} \\ &\qquad + \frac{-q^2}{(q-q^{-1})^2}F_iE_i. \end{aligned}$$

There is a term in Y_7 (cf. (3.17)):

$$-q^{-2}F_iE_iP_j = -\frac{-q^{-2}}{[2]^2}F_iE_i\frac{(1+q^{-2})K_{2j} + (1+q^2)K_{-2j}}{(q-q^{-1})^2} + \frac{q^{-2}}{(q-q^{-1})^2}F_iE_i.$$

So the summand containing F_iE_i in $[T_j(E_i), T_j(F_i)]$ is given by

$$\begin{aligned} &-F_iP_jE_i - qQ_jF_iE_ix_j - q^2P_jF_iE_i + q^2Q_jF_iE_iQ_j - q^{-2}F_iE_iP_j \\ &\quad = f_1(q)F_iE_iK_{2j} + f_2(q)F_iE_iK_{-2j} + \frac{2+q^{-2}}{(q-q^{-1})^2}F_iE_i, \end{aligned}$$

where

$$f_1(q) = -\frac{1}{[2]^2}\big(q^{-4} + q^2 + 2q^{-2} + 2\big), \tag{3.21}$$

$$f_2(q) = -\frac{1}{[2]^2}\big(q^{-4} + q^4 + 2q^{-2} + q^2 + 1\big). \tag{3.22}$$

(ii) Consider the coefficient of E_iF_i in $[T_j(E_i), T_j(F_i)]$. There are terms in Y_9 (cf. (3.19)) and Y_6 (cf. (3.16)): $-E_iP_jF_i + E_iQ_j^2F_i$ and $q^{-1}E_iF_iQ_j\bar{x}_j$. By a completely similar computation as above it follows that the summand containing E_iF_i in $[T_j(E_i), T_j(F_i)]$ is given by

$$\begin{aligned}
&-E_iP_jF_i + E_iQ_j^2F_i + q^{-1}E_iF_iQ_j\bar{x}_j \\
&\quad = -f_1(q)F_iE_iK_{2j} + f_2(q)F_iE_iK_{-2j} - \frac{2+q^{-2}}{(q-q^{-1})^2}F_iE_i,
\end{aligned}$$

where $f_1(q)$ is given by (3.21) and $f_2(q)$ is given by (3.22), respectively.

So, there is a summand in $[T_j(E_i), T_j(F_i)]$ as follows.

$$\begin{aligned}
&(F_iE_i - E_iF_i)\left\{f_1(q)K_{2j} + f_2(q)K_{2j}^{-1} + \frac{2+q^{-2}}{(q-q^{-1})^2}\right\} \\
&\quad = -Q_i\left[f_1(q)K_{2j} + f_2(q)K_{-2j} + \frac{2+q^{-2}}{(q-q^{-1})^2}\right] \\
&\quad = -\frac{f_1(q)}{q-q^{-1}}K_{i+2j} + \frac{f_2(q)}{q-q^{-1}}K_{-i-2j} - \frac{f_2(q)}{q-q^{-1}}K_{i-2j} \\
&\qquad + \frac{f_1(q)}{q-q^{-1}}K_{-i+2j} - \frac{2+q^{-2}}{(q-q^{-1})^3}K_i + \frac{2+q^{-2}}{(q-q^{-1})^3}K_{-i}.
\end{aligned} \tag{3.23}$$

(iii) Similarly, by Y_3 and Y_8 the summands of $[T_j(E_i), T_j(F_i)]$ containing E_jF_j is given by

$$\begin{aligned}
&qE_jF_jQ_ix_j + E_jQ_iF_jQ_j \\
&\quad = E_jF_j\big\{g_1(q)K_{i+j} + g_2(q)K_{i+j}^{-1} + g_3(q)K_{i-j} + g_4(q)K_{-i+j}\big\},
\end{aligned}$$

where

$$g_1(q) = \frac{1+q^{-2}}{(q-q^{-1})^2}, \qquad g_2(q) = \frac{2q^2}{(q-q^{-1})^2}, \tag{3.24}$$

$$g_3(q) = -\frac{q^2+q^{-2}}{(q-q^{-1})^2}, \qquad g_4(q) = -\frac{1+q^2}{(q-q^{-1})^2}. \tag{3.25}$$

(iv) By Y_8, Y_7, Y_4, Y_9, Y_5, Y_6 the summands of $[T_j(E_i), T_j(F_i)]$ containing F_jE_j is given by

$$
\begin{aligned}
&-F_jQ_iE_jQ_j+F_jQ_jE_jQ_i+Q_jF_jE_jQ_i+q^2Q_iF_jE_jQ_j-q^{-1}Q_iF_jE_j\bar{x}_j\\
&\quad+F_jE_jQ_j\frac{q^{-4}K_i-q^4K_i^{-1}}{q-q^{-1}}-q^{-1}\big(q+q^{-1}\big)F_jE_jQ_j\frac{q^{-2}K_i-q^2K_i^{-1}}{q-q^{-1}}\\
&\quad-q\big(q+q^{-1}\big)F_jE_jQ_j\frac{q^{-2}K_i-q^2K_i^{-1}}{q-q^{-1}}\\
&=F_jE_j\big\{-g_1(q)K_{i+j}-g_2(q)K_{i+j}^{-1}-g_3(q)K_{i-j}-g_4(q)K_{-i+j}\big\},
\end{aligned}
$$

where $g_1(q), g_2(q), g_3(q), g_4(q)$ are given by (3.24), (3.25), respectively.

It follows that there is a summand in $[T_j(E_i), T_j(F_i)]$ as follows.

$$
\begin{aligned}
&(E_jF_j-F_jE_j)\big\{g_1(q)K_{i+j}+g_2(q)K_{i+j}^{-1}+g_3(q)K_{i-j}+g_4(q)K_{-i+j}\big\}\\
&\quad=Q_j\big[g_1(q)K_{i+j}+g_2(q)K_{-i-j}+g_3(q)K_{i-j}+g_4(q)K_{-i+j}\big]\\
&\quad=\frac{g_1(q)}{q-q^{-1}}K_{i+2j}-\frac{g_2(q)}{q-q^{-1}}K_{-i-2j}-\frac{g_3(q)}{q-q^{-1}}K_{i-2j}\\
&\qquad+\frac{g_4(q)}{q-q^{-1}}K_{-i+2j}+\frac{g_3(q)-g_1(q)}{q-q^{-1}}K_i+\frac{g_2(q)-g_4(q)}{q-q^{-1}}K_{-i}.
\end{aligned}\tag{3.26}
$$

(v) From Y_3 (cf. (3.13)), Y_4 (cf. (3.14)), Y_5 (cf. (3.15)), Y_9 (cf. (3.19)), the remaining summand containing no letters E_i, E_j, F_i, F_j in $[T_j(E_i), T_j(F_i)]$ is:

$$
\begin{aligned}
&-q\big(q+q^{-1}\big)P_j\frac{q^{-2}K_i-q^2K_{-i}}{q-q^{-1}}-qQ_iQ_jx_j\\
&\quad-q^{-1}Q_iQ_j\bar{x}_j-q^{-1}\big(q+q^{-1}\big)P_j\frac{q^{-2}K_i-q^2K_{-i}}{q-q^{-1}}+P_j\frac{q^{-4}K_i-q^4K_{-i}}{q-q^{-1}}\\
&=h_1(q)K_{i+2j}+h_2(q)K_{-i-2j}+h_3(q)K_{i-2j}\\
&\quad+h_4(q)K_{-i+2j}+h_5(q)K_i+h_6(q)K_{-i},
\end{aligned}\tag{3.27}
$$

where

$$h_1(q)=-\frac{2q^{-4}+2q^2+5q^{-2}+5}{[2]^2(q-q^{-1})^3},$$

$$h_2(q)=\frac{3q^4+q^{-4}+5q^2+2q^{-2}+3}{[2]^2(q-q^{-1})^3},$$

$$h_3(q)=-\frac{q^{-4}+q^4+3q^2+4q^{-2}+5}{[2]^2(q-q^{-1})^3},$$

$$h_4(q)=\frac{4q^2+3q^{-2}+7}{[2]^2(q-q^{-1})^3},\qquad h_5(q)=\frac{q^2+3q^{-2}+3}{(q-q^{-1})^3},\qquad h_6(q)=-\frac{3q^2+q^{-2}+3}{(q-q^{-1})^3}.$$

So, in the sum (3.23) + (3.26) + (3.27), the coefficient of K_i is

$$\frac{g_3(q) - g_1(q)}{q - q^{-1}} + h_5(q) - \frac{2 + q^{-2}}{(q - q^{-1})^3} = 0;$$

the coefficient of K_{-i} is

$$\frac{g_2(q) - g_4(q)}{q - q^{-1}} + h_6(q) + \frac{2 + q^{-2}}{(q - q^{-1})^3} = 0;$$

the coefficient of K_{-i+2j} is

$$\frac{g_4(q)}{q - q^{-1}} + f_1(q) + h_4(q) = -\frac{1}{q - q^{-1}};$$

and the coefficient of K_{i-2j} is

$$\frac{-f_2(q)}{q - q^{-1}} - \frac{g_3(q)}{q - q^{-1}} + h_3(q) = \frac{1}{q - q^{-1}}.$$

So, by (3.20) it follows that $[T_j(E_i), T_j(F_i)] = (K_{i+2j} - K_{-i-2j})/(q - q^{-1})$ as required. This completes the proof of Lemma 3.4. $\square$

Up to now we have shown that T_j respects the relation (Re. 1).

4. The affine Serre relations

In this section we show that for any $1 \leqslant k \leqslant \ell + 2$, T_k respects the defining relation (Re. 2) of U_q: For $a_{ij} \leqslant 0$,

$$[E_i, F_j] = 0, \tag{4.1}$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s E_i^{(1-a_{ij}-s)} E_j E_i^{(s)} = 0, \tag{4.2}$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s F_i^{(1-a_{ij}-s)} F_j F_i^{(s)} = 0. \tag{4.3}$$

It suffices to check that T_k respects (4.1) and (4.2). Assume that $a_{ij} \leqslant 0$. It can be verified directly that $[T_k(E_i), T_k(F_j)] = 0$ holds in the following cases:

(i) $k = i \neq j$;
(ii) $k = j \neq i$;
(iii) $k \neq i$, $k \neq j$, $a_{ki} \leqslant 0$, $a_{kj} \leqslant 0$.

Since the case $a_{ki} = a_{kj} = 2$ with $k \neq i, k \neq j$ does not occur in our setting (we consider the 2-affinization), to show that T_k respects the defining relation (4.1), it suffices to consider the following cases:

(iv) $k \neq i, k \neq j, a_{ki} = 2$;
(v) $k \neq i, k \neq j, a_{kj} = 2$.

Now we consider the case (iv) since (v) is similar. If $a_{kj} = 0$, then by the definitions (2.6), (2.7) and the Serre relation (2.3), it follows that

$$\big[T_k(E_i), T_k(F_j)\big] = q^2\big[F_k^{(2)}E_i - q^{-1}F_kE_iF_k + q^{-2}E_iF_k^{(2)}, F_j\big] = 0.$$

So, to prove that T_k respect the relation (4.1), it suffices to show the following

Proposition 4.1. *Assume that $a_{ij} \leqslant 0$, $a_{kj} = -1$ and $a_{ki} = 2$ but $k \neq i$. Then $[T_k(E_i), T_k(F_j)] = 0$.*

Proof. By (2.6), (2.7) it follows that $[T_k(E_i), T_k(F_j)] = q^2\sum_{t=1}^{12} Y_t$, where

$$\begin{aligned} Y_1 &= -F_k^{(2)}E_iF_jF_k, & Y_2 &= q^{-1}F_kE_iF_kF_jF_k,\\ Y_3 &= -q^{-2}E_iF_k^{(2)}F_jF_k, & Y_4 &= [3]F_jF_k^{(3)}E_i,\\ Y_5 &= -q^{-1}F_jF_k^2E_iF_k, & Y_6 &= q^{-2}F_jF_kE_iF_k^{(2)},\\ Y_7 &= qF_k^{(2)}E_iF_kF_j, & Y_8 &= -F_kE_iF_k^2F_j,\\ Y_9 &= q^{-1}[3]E_iF_k^{(3)}F_j, & Y_{10} &= -qF_kF_jF_k^{(2)}E_i,\\ Y_{11} &= F_kF_jF_kE_iF_k, & Y_{12} &= -q^{-1}F_kF_jE_iF_k^2. \end{aligned}$$

Since $a_{kj} = -1$, by (2.1) it follows that $Y_1 + Y_{11} = F_jF_k^{(2)}E_iF_k$. Hence

$$\begin{aligned} Y_1 + Y_{11} + Y_5 + Y_6 &= F_jF_k^{(2)}E_iF_k - [2]q^{-1}F_jF_k^{(2)}E_iF_k + q^{-2}F_jF_kE_iF_k^{(2)}\\ &= q^{-2}F_j\big(-F_k^{(2)}E_iF_k + F_kE_iF_k^{(2)}\big)\\ &= q^{-2}F_jE_iF_k^{(3)} - q^{-2}F_jF_k^{(3)}E_i. \end{aligned} \tag{4.4}$$

In the last equality we used (2.2). Similarly, $Y_2 + Y_{12} = q^{-1}F_k\ E_iF_k^{(2)}F_j$. Hence

$$Y_2 + Y_{12} + Y_8 + Y_7 = qF_k^{(3)}E_iF_j - qE_iF_k^{(3)}F_j. \tag{4.5}$$

By the Serre relation (2.1), we have that

$$
\begin{aligned}
Y_3 &= -q^{-2}E_iF_k^{(2)}F_jF_k = -q^{-2}E_i\big(F_kF_jF_k - F_jF_k^{(2)}\big)F_k \\
&= -q^{-2}[2]E_iF_k(F_kF_jF_k - F_k^{(2)}F_j) + [3]q^{-2}E_iF_jF_k^{(3)} \\
&= -q^{-2}[2]^2E_iF_k^{(2)}F_jF_k + [2][3]q^{-2}E_iF_k^{(3)}F_j + [3]q^{-2}E_iF_jF_k^{(3)} \\
&= [2]^2Y_3 + [2][3]q^{-2}E_iF_k^{(3)}F_j + [3]q^{-2}E_iF_jF_k^{(3)}.
\end{aligned}
$$

Since $1-[2]^2 = -[3]$, it follows that

$$
Y_3 = -[2]q^{-2}E_iF_k^{(3)}F_j - q^{-2}E_iF_jF_k^{(3)}. \tag{4.6}
$$

By a completely similar computation we have that

$$
Y_{10} = -[2]qF_jF_k^{(3)}E_i - qF_k^{(3)}F_jE_i. \tag{4.7}
$$

By (4.4), (4.5), (4.6), (4.7) and the fact that $E_iF_j = F_jE_i$ it follows that

$$
\begin{aligned}
&\big[T_k(E_i), T_k(F_j)\big] \\
&\quad = q^2\big[\big(q^{-2}-q^{-2}\big)E_iF_jF_k^{(3)} + \big([3]q^{-1} - q - [2]q^{-2}\big)E_iF_k^{(3)}F_j \\
&\qquad + \big(-q^{-2} - q[2] + [3]\big)F_jF_k^{(3)}E_i + (q-q)F_k^{(3)}E_iF_j\big] = 0.
\end{aligned}
$$

This completes the proof. □

Now we show that T_k $(1 \leqslant k \leqslant \ell+2)$ respects the Serre relation (4.2). At first we consider the case either $k=i$ or $k=j$.

Proposition 4.2. *Assume that $a_{ij} \leqslant 0$. Then we have*

$$
\sum_{s=0}^{1-a_{ij}} (-1)^s\big(T_i(E_i)\big)^{(1-a_{ij}-s)}T_i(E_j)\big(T_i(E_i)\big)^{(s)} = 0, \tag{4.8}
$$

$$
\sum_{s=0}^{1-a_{ij}} (-1)^s\big(T_j(E_i)\big)^{(1-a_{ij}-s)}T_j(E_j)\big(T_j(E_i)\big)^{(s)} = 0. \tag{4.9}
$$

Proof. It is easy to see that (4.8) and (4.9) hold if $a_{ij}=0$. Now assume that $a_{ij}=-1$. Then (4.8) and (4.9) follows directly by the following facts, which can be verified directly:

$$
\begin{aligned}
T_i(E_i)T_i(E_j) &= qT_i(E_j)T_i(E_i) + qE_j, \\
T_j(E_i)T_j(E_j) &= q^{-1}T_j(E_j)T_j(E_i) - q^{-1}E_i.
\end{aligned}
$$

This completes the proof. □

Now we consider the case that $k \neq i$, $k \neq j$ and $a_{ij} = 0$. Note that in this case the Serre relation (4.2) becomes $E_i E_j - E_j E_i = 0$.

Proposition 4.3. *Assume that $k \neq i$, $k \neq j$ and $a_{ij} = 0$. Then*

$$T_k(E_i)T_k(E_j) - T_k(E_j)T_k(E_i) = 0. \tag{4.10}$$

Proof. We consider the following cases. It is easy to see that (4.10) holds in cases (i), (ii), (iii).

(i) $a_{ki} = a_{kj} = 0$.
(ii) $a_{ki} = -1, a_{kj} = 0$.
(iii) $a_{ki} = 0, a_{kj} = -1$.
(iv) $a_{ki} = a_{kj} = -1$. Then, by the Serre relation (4.2) it follows that

$$-q^{-1}E_i E_k^2 E_j + q^{-2}E_i E_k E_j E_k = -E_i E_k^{(2)} E_j + q^{-2}E_i E_j E_k^{(2)},$$
$$q^{-1}E_j E_k^2 E_i - q^{-2}E_j E_k E_i E_k = E_j E_k^{(2)} E_i + q^{-2}E_j E_i E_k^{(2)}.$$

So,

$$\begin{aligned}
&T_k(E_i)T_k(E_j) - T_k(E_j)T_k(E_i)\\
&\quad = E_k E_i E_k E_j - q^{-1}E_i E_k^2 E_j + q^{-2}E_i E_k E_j E_k\\
&\qquad - E_k E_j E_k E_i + q^{-1}E_j E_k^2 E_i - q^{-2}E_j E_k E_i E_k\\
&\quad = E_k E_i E_k E_j - E_i E_k^{(2)} E_j - E_k E_j E_k E_i + E_j E_k^{(2)}\\
&\quad = E_k^{(2)} E_i E_j - E_k^{(2)} E_j E_i = 0
\end{aligned}$$

and (4.10) follows.

(v) $a_{ki} = 2, a_{kj} = 0$. By the definition (2.6) of T_k we have that

$$\begin{aligned}
&T_k(E_i)T_k(E_j) - T_k(E_j)T_k(E_i)\\
&\quad = q^2\big[F_k^{(2)} E_i E_j - q^{-1}F_k E_i F_k E_j + q^{-2}E_i F_k^{(2)} E_j\\
&\qquad - E_j F_k^{(2)} E_i + q^{-1}E_j F_k E_i F_k - q^{-2}E_j E_i F_k^{(2)}\big] = 0,
\end{aligned}$$

noting that $E_i E_j = E_j E_i$ and $F_k E_j = E_j F_k$. So (4.10) follows.

(vi) $a_{ki} = 2$, $a_{kj} = -1$. Note that in this case we have that $E_k E_i = E_i E_k$ by (2.3) and $E_j F_k = F_k E_j$. Also, $E_i E_j = E_j E_i$. By the definition (2.6) of T_k we have that $T_k(E_i)T_k(E_j) - T_k(E_j)T_k(E_i) = q^2 \sum_{t=1}^{12} M_t$, where

$$M_1 = -F_k^{(2)} E_i E_k E_j = -F_k^{(2)} E_k E_i E_j,$$
$$M_2 = q^{-1}F_k^{(2)} E_i E_j E_k = q^{-1}E_j F_k^{(2)} E_k E_i,$$
$$M_3 = q^{-1}F_k E_i F_k E_k E_j = q^{-1}F_k E_k E_i E_j F_k - q^{-1}F_k E_i Q_k E_j,$$

$$
\begin{aligned}
M_4 &= -q^{-2}F_kE_iF_kE_jE_k = -q^{-2}F_kE_iE_jE_kF_k + q^{-2}F_kE_iE_jQ_k,\\
M_5 &= -q^{-2}E_iF_k^{(2)}E_kE_j,\\
M_6 &= q^{-3}E_iF_k^{(2)}E_jE_k = q^{-3}E_iE_jF_k^{(2)}E_k,\\
M_7 &= E_kE_jF_k^{(2)}E_i = E_kF_k^{(2)}E_iE_j\\
&= F_k^{(2)}E_kE_iE_j + F_kx_kE_iE_j \quad \text{(by Lemma 3.3)},\\
M_8 &= -q^{-1}E_kE_jF_kE_iF_k = -q^{-1}F_kE_kE_jE_iF_k - q^{-1}Q_kE_iE_jF_k,\\
M_9 &= q^{-2}E_kE_jE_iF_k^{(2)} = q^{-2}E_iE_kF_k^{(2)}E_j\\
&= q^{-2}E_iF_k^{(2)}E_kE_j + q^{-2}E_iF_kx_kE_j \quad \text{(by Lemma 3.3)},\\
M_{10} &= -q^{-1}E_jE_kF_k^{(2)}E_i\\
&= -q^{-1}E_jF_k^{(2)}E_kE_i - q^{-1}E_jF_kx_kE_i \quad \text{(by Lemma 3.3)},\\
M_{11} &= q^{-2}E_jE_kF_kE_iF_k = q^{-2}F_kE_jE_iE_kF_k + q^{-2}E_jQ_kE_iF_k,\\
M_{12} &= -q^{-3}E_jE_kE_iF_k^{(2)} = -q^{-3}E_iE_jF_k^{(2)}E_k - q^{-3}E_iE_jF_kx_k.
\end{aligned}
$$

Note that $K_kE_i = q^2E_iK_k$, $K_kE_j = q^{-1}E_jK_k$. It follows that

$$
\begin{aligned}
M_1 + M_7 &= F_kx_kE_iE_j = F_kE_iE_j\frac{(q+q^{-1})K_k - (q+q^{-1})K_k^{-1}}{q^2-q^{-2}},\\
M_2 + M_{10} &= -q^{-1}E_jF_kx_kE_i\\
&= -F_kE_iE_j\frac{(q+q^{-1})K_k - (q^{-3}+q^{-1})K_k^{-1}}{q^2-q^{-2}},\\
M_3 + M_8 &= -q^{-1}F_kE_iQ_kE_j - q^{-1}Q_kE_iE_jF_k\\
&= -F_kE_iE_j\frac{q^{-2}K_k - K_k^{-1}}{q-q^{-1}} - E_iE_jF_k\frac{q^{-2}K_k - K_k^{-1}}{q-q^{-1}},\\
M_4 + M_{11} &= q^{-2}F_kE_iE_jQ_k + q^{-2}E_jQ_kE_iF_k\\
&= F_kE_iE_j\frac{q^{-2}K_k - q^{-2}K_k^{-1}}{q-q^{-1}} + E_jE_iF_k\frac{q^{-2}K_k - q^{-2}K_k^{-1}}{q-q^{-1}},\\
M_5 + M_9 &= q^{-2}E_iF_kx_kE_j\\
&= E_iE_jF_k\frac{(q^{-3}+q^{-5})K_k - (q+q^{-1})K_k^{-1}}{q^2-q^{-2}},\\
M_6 + M_{12} &= -q^{-3}E_iE_jF_kx_k\\
&= -E_iE_jF_k\frac{(q^{-3}++q^{-5})K_k - (q^{-3}+q^{-1})K_k^{-1}}{q^2-q^{-2}}.
\end{aligned}
$$

Summing these formulae up, we see that $\sum_{t=1}^{12} M_t = 0$ and (4.10) follows.

(vii) $a_{kj} = 2$, $a_{ki} = 0$. This is similar to (v).

(viii) $a_{kj} = 2$, $a_{ki} = -1$. This is similar to (vi).

This completes the proof. $\square$

Now we consider the case that $k \neq i$, $k \neq j$ but $a_{ij} = -1$. In this case the Serre relation (4.2) becomes

$$E_i^{(2)} E_j - E_i E_j E_i + E_j E_i^{(2)} = 0.$$

We shall show that the following equality holds:

$$\big(T_k(E_i)\big)^{(2)} T_k(E_j) - T_k(E_i) T_k(E_j) T_k(E_i) + T_k(E_j) \big(T_k(E_i)\big)^{(2)} = 0. \tag{4.11}$$

Lemma 4.1. *The equality* (4.11) *holds in the following cases*:

(i) $a_{ki} = a_{kj} = 0$.
(ii) $a_{ki} = -1$, $a_{kj} = 0$.
(iii) $a_{ki} = 0$, $a_{kj} = -1$.

Proof. The case (i) is clear. It suffices to prove (4.11) for the case (ii) since the case (iii) is obtained by interchanging the indices i, j from the case (ii).

Assume that $a_{ki} = -1$, $a_{kj} = 0$. Note that in this case we have that

$$T_k(E_i) = -E_k E_i + q^{-1} E_i E_k, \qquad T_k(E_j) = E_j, \qquad E_k E_j = E_j E_k.$$

It follows that

$$\begin{aligned}
&\big(T_k(E_i)\big)^{(2)} T_k(E_j) - T_k(E_i) T_k(E_j) T_k(E_i) + T_k(E_j) \big(T_k(E_i)\big)^{(2)} \\
&\quad = \frac{1}{[2]} \big\{ \big(T_k(E_i)\big)^2 E_j - \big(q + q^{-1}\big) T_k(E_i) E_j T_k(E_i) + E_j \big(T_k(E_i)\big)^2 \big\}.
\end{aligned} \tag{4.12}$$

Note that, by the Serre relation (4.2), $(T_k(E_i))^2 E_j = X_{11} + X_{12} + X_{13} + X_{14}$, where

$$\begin{aligned}
X_{11} &= E_k E_i E_k E_i E_j = E_k^{(2)} E_i^2 E_j + E_i E_k^{(2)} E_i E_j, \\
X_{12} &= -q^{-1} E_k E_i^2 E_k E_j, \\
X_{13} &= -q^{-1} E_i E_k^2 E_i E_j = -\big(1 + q^2\big) E_i E_k^{(2)} E_i E_j, \\
X_{14} &= q^{-2} E_i E_k E_i E_k E_j = q^{-2} E_i E_k^{(2)} E_i E_j + q^{-2} E_i^2 E_j E_k^{(2)},
\end{aligned}$$

and $-(q + q^{-1}) T_k(E_i) E_j T_k(E_i) = X_{21} + X_{22} + X_{23} + X_{24}$, where

$$\begin{aligned}
X_{21} &= -\left(q+q^{-1}\right)E_kE_iE_jE_kE_i \\
&= -E_k^{(2)}E_i^2E_j - E_jE_k^{(2)}E_i^2 - \left(q+q^{-1}\right)E_iE_k^{(2)}E_jE_i, \\
X_{22} &= q^{-1}\left(q+q^{-1}\right)E_iE_kE_jE_kE_i = (q+q^{-1})^2q^{-1}E_iE_k^{(2)}E_jE_i, \\
X_{23} &= \left(q+q^{-1}\right)E_iE_kE_jE_kE_i = q^{-1}(q+q^{-1})^2E_iE_k^{(2)}E_jE_i, \\
X_{24} &= -q^{-2}\left(q+q^{-1}\right)E_iE_kE_jE_iE_k \\
&= -q^{-2}\left(q+q^{-1}\right)E_iE_jE_k^{(2)}E_i - q^{-2}E_jE_i^2E_k^{(2)} - q^{-2}E_i^2E_jE_k^{(2)},
\end{aligned}$$

and $E_j(T_k(E_i))^2 = X_{31}+X_{32}+X_{33}+X_{34}$, where

$$\begin{aligned}
X_{31} &= E_jE_kE_iE_kE_i = E_jE_k^{(2)}E_i^2E_i + E_jE_iE_k^{(2)}E_i, \\
X_{32} &= -q^{-1}E_jE_iE_k^2E_i = -\left(1+q^{-2}\right)E_jE_iE_k^{(2)}E_i, \\
X_{33} &= -q^{-1}E_jE_iE_k^2E_i = -\left(1+q^{-2}\right)E_jE_iE_k^{(2)}E_i, \\
X_{34} &= q^{-2}E_jE_iE_kE_iE_k = q^{-2}E_jE_iE_k^{(2)}E_i + q^{-2}E_jE_i^2E_k^{(2)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&X_{12}+X_{22}+X_{32} = 0, \\
&X_{11}+X_{21}+X_{31} = -\left(q+q^{-1}\right)E_iE_k^{(2)}E_jE_i + E_iE_k^{(2)}E_iE_j + E_jE_iE_k^{(2)}E_i, \\
&X_{13}+X_{23}+X_{33} = \left(q+q^{-1}\right)^2q^{-1}E_iE_k^{(2)}E_jE_i - \left(1+q^{-2}\right)E_iE_k^{(2)}E_iE_j \\
&\qquad\qquad - \left(1+q^{-2}\right)E_jE_iE_k^{(2)}E_i, \\
&X_{14}+X_{24}+X_{34} = \left(-q^{-2}\left(q+q^{-1}\right)\right)E_iE_jE_k^{(2)}E_i \\
&\qquad\qquad + q^{-2}E_iE_k^{(2)}E_iE_j + q^{-2}E_jE_iE_k^{(2)}E_i.
\end{aligned}$$

Since $E_kE_j = E_jE_k$, it follows that

$$\begin{aligned}
&X_{11}+X_{21}+X_{31}+X_{13}+X_{23}+X_{33}+X_{14}+X_{24}+X_{34} \\
&\quad = \left(-\left(q+q^{-1}\right)+\left(q+q^{-1}\right)^2q^{-1}-q^{-2}\left(q+q^{-1}\right)\right)E_iE_jE_k^{(2)}E_i = 0,
\end{aligned}$$

and hence $(T_k(E_i))^2E_j - (q+q^{-1})T_k(E_i)E_jT_k(E_i) + E_j(T_k(E_i))^2 = 0$. Then (4.11) follows by (4.12). This completes the proof. □

Proposition 4.4. *Assume that $a_{ki} = a_{kj} = a_{ij} = -1$. Then* (4.11) *holds.*

We need the following lemmas, which will be used in next sections.

Lemma 4.2. *Assume that $a_{ki} = -1$. Then we have the following equalities*:

(i) $E_kT_k(E_i) = qT_k(E_i)E_k$;
(ii) $E_iT_k(E_i) = q^{-1}T_k(E_i)E_i$;
(iii) $F_kT_k(E_i) = T_k(E_i)F_k - E_iK_k^{-1}$;
(iv) $K_kT_k(E_i) = qT_k(E_i)K_k$.

Proof. (i) and (ii) follows directly by the Serre relation (4.2); while (iii) follows directly by the facts that $E_iF_k = F_kE_i$ and $[E_k, F_k] = Q_k$, and (iv) follows by $K_kE_k = q^2E_kK_k$ and $K_kE_i = q^{-1}E_iK_k$. □

Proof of Proposition 4.4. For brevity, set

$$\begin{aligned}\Theta = \big(T_k(E_i)\big)^2T_k(E_j) - \big(q+q^{-1}\big)T_k(E_i)T_k(E_j)T_k(E_i)\\ + T_k(E_j)\big(T_k(E_i)\big)^2.\end{aligned} \tag{4.13}$$

Then, to show (4.11), it suffices to show that $\Theta = 0$. Then by Lemma 4.2, it follows that

$$E_k\Theta = q^3\Theta E_k, \qquad K_k\Theta = q^3\Theta K_k. \tag{4.14}$$

Also by Lemma 4.2, it follows that

$$\big(T_k(E_i)\big)^2F_k = F_k\big(T_k(E_i)\big)^2 + \big(1+q^{-2}\big)T_k(E_i)E_iK_k^{-1}. \tag{4.15}$$

Then, by a lengthy but straightforward computation we have that

$$\Theta F_k = F_k\Theta + (\Theta_1 + \Theta_2)K_k^{-1}, \tag{4.16}$$

where

$$\begin{aligned}\Theta_1 &= \big(T_k(E_i)\big)^2E_j - \big(q+q^{-1}\big)q^{-1}T_k(E_i)E_jT_k(E_i) + q^{-2}E_j\big(T_k(E_i)\big)^2,\\ \Theta_2 &= \big(1+q^{-2}\big)E_iT_k(E_i)T_k(E_j) - \big(q+q^{-1}\big)q^{-2}E_iT_k(E_j)T_k(E_i)\\ &\quad + \big(1+q^{-2}\big)T_k(E_j)T_k(E_i)E_i - \big(q+q^{-1}\big)T_k(E_i)T_k(E_j)E_i.\end{aligned}$$

In a similar way as above using Lemma 4.2 and (4.15), we have that $F_k(\Theta_1 + \Theta_2) = (\Theta_1 + \Theta_2)F_k + \Theta_3K_k^{-1}$, where

$$\begin{aligned}\Theta_3 &= -q^{-1}\big(q+q^{-1}\big)^2T_k(E_i)E_iE_j + \big(q+q^{-1}\big)^2T_k(E_i)E_jE_i\\ &\quad + \big(q+q^{-1}\big)\big(q^{-1}+q^{-3}\big)E_iE_jT_k(E_i) - q^{-1}\big(q+q^{-1}\big)^2E_jE_iT_k(E_i)\\ &\quad - \big(1+q^{-2}\big)q^{-1}E_i^2T_k(E_j) + q^{-1}\big(q+q^{-1}\big)^2E_iT_k(E_j)E_i\\ &\quad - \big(q+q^{-1}\big)T_k(E_j)E_i^2.\end{aligned}$$

In each summand of Θ_3 we move the letter E_k properly by using the Serre relation (4.2) to obtain that

$$\begin{aligned}\Theta_3 = {} & \big(q+q^{-1}\big)E_k\big(E_i^2E_j-\big(q+q^{-1}\big)E_iE_jE_i+E_jE_i^2\big)\\ & -\big(1+q^{-2}\big)^{-2}\big(E_i^2E_j-\big(q+q^{-1}\big)E_iE_jE_i+E_jE_i^2\big)E_k=0,\end{aligned}$$

which means that

$$F_k(\Theta_1+\Theta_2)=(\Theta_1+\Theta_2)F_k. \tag{4.17}$$

On the other hand, by using Lemma 4.2, we have that

$$\begin{aligned}E_k\Theta_1-q^3\Theta_1E_k &= -q^2\big(T_k(E_i)\big)^2T_k(E_j)-q^{-2}T_k(E_j)\big(T_k(E_i)\big)^2\\ &\quad+\big(q+q^{-1}\big)T_k(E_i)T_k(E_j)T_k(E_i),\\ E_k\Theta_2-q^3\Theta_2E_k &= -[2]q^{-1}\big(T_k(E_i)\big)^2T_k(E_j)-[2]qT_k(E_j)\big(T_k(E_i)\big)^2\\ &\quad+[2][3]T_k(E_i)T_k(E_j)T_k(E_i).\end{aligned}$$

So we have that

$$E_k\Theta_1-q^3\Theta_1E_k+E_k\Theta_2-q^3\Theta_2E_k=-[3]\Theta. \tag{4.18}$$

Note that, by (4.14) and (4.16), it follows that

$$\begin{aligned}q^3\Theta E_kF_k &= E_kF_k\Theta+(E_k\Theta_1+E_k\Theta_2)K_k^{-1},\\ q^3\Theta F_kE_k &= F_kE_k\Theta+(q\Theta_1E_k+q\Theta_2E_k)K_k^{-1}.\end{aligned}$$

It follows that $q^3\Theta Q_k=Q_k\Theta+[E_k\Theta_1+E_k\Theta_2-q\Theta_1E_k-q\Theta_2E_k]K_k^{-1}$, i.e.,

$$E_k\Theta_1+E_k\Theta_2-q\Theta_1E_k-q\Theta_2E_k=-[3]\Theta. \tag{4.19}$$

Comparing (4.18) and (4.19), we have that $(\Theta_1+\Theta_2)E_k=0$. By (4.17) it follows that $(\Theta_1+\Theta_2)F_kE_k=0=(\Theta_1+\Theta_2)E_kF_k$. So $(\Theta_1+\Theta_2)Q_k=0$ and hence $\Theta_1+\Theta_2=0$. thus, by (4.18) it follows that $\Theta=0$ as required. This completes the proof. □

We consider further the remaining cases.

Proposition 4.5. *Assume that $a_{ki}=2$, $a_{kj}=0$ and $a_{ij}=-1$. Then the equality* (4.11) *holds.*

For brevity we modify the definition of T_k as follows (cf. (2.6)). Put

$$\widehat{T}_k(E_i) = F_k^{(2)} E_i - q^{-1} F_k E_i F_k + q^{-2} E_i F_k^{(2)}, \qquad \widehat{T}_k(E_j) = E_j,$$
$$\Lambda = \big(\widehat{T}_k(E_i)\big)^2 E_j - \big(q + q^{-1}\big)\widehat{T}_k(E_i) E_j \widehat{T}_k(E_i) + E_j\big(\widehat{T}_k(E_i)\big)^2. \tag{4.20}$$

Then (4.11) is equivalent to $\Lambda = 0$. Introduce

$$\varphi = F_k E_i - q^{-2} E_i F_k. \tag{4.21}$$

Then for $a_{ki} = 2$ but $k \neq i$,

$$\widehat{T}_k(E_i) = \frac{1}{[2]}(F_k\varphi - \varphi F_k). \tag{4.22}$$

We also have the following

Lemma 4.3. *Assume that $a_{ki} = 2$ but $k \neq i$. Then*

(i) $K_k\varphi = \varphi K_k$;
(ii) $F_k^2\varphi - (1+q^2)F_k\varphi F_k + q^2\varphi F_k^2 = 0$;
(iii) $E_k\varphi = \varphi E_k + [2]E_i K_k$;
(iv) $E_k\varphi^2 = \varphi^2 E_k + [2](\varphi E_i + E_i\varphi)K_k$.

Proof. (i) is clear. (ii) follows by the Serre relation (2.2). (iii) follows by the facts that $[E_k, F_k] = Q_k$ and $E_i E_k = E_k E_i$. (iv) follows by (iii). □

Lemma 4.4. *Assume that $a_{ki} = 2$. Then*

(i) $F_k\widehat{T}_k(E_i) = q^2\widehat{T}_k(E_i)F_k$;
(ii) $E_k\widehat{T}_k(E_i) = \widehat{T}_k(E_i)E_k + \varphi K_k$;
(iii) $E_k(\widehat{T}_k(E_i))^2 = (\widehat{T}_k(E_i))^2 E_k + [\widehat{T}_k(E_i)\varphi + q^{-2}\varphi\widehat{T}_k(E_i)]K_k$.

Proof. (i) follows by (ii) of Lemma 4.3. (ii) follows by (4.22) and (iii) of Lemma 4.3, (iii) follows by (ii). □

Proof of Proposition 4.5. Since $a_{kj} = 0$ it follows that $F_k E_j = E_j F_k$. Thus, by (i) of Lemma 4.4 it follows that

$$F_k\Lambda = q^4\Lambda F_k. \tag{4.23}$$

On the other hand, by a direct computation using (ii) and (iii) of Lemma 4.4 we have that

$$E_k\Lambda = \Lambda E_k + \Lambda_1 K_k, \tag{4.24}$$

where

$$\begin{aligned}\Lambda_1 = {}& \widehat{T}_k(E_i)\varphi E_j - \big(q+q^{-1}\big)q^{-2}\varphi E_j\widehat{T}_k(E_i) + q^{-2}E_j\varphi\widehat{T}_k(E_i)\\ &+ q^{-2}\varphi\widehat{T}_k(E_i)E_j - \big(q+q^{-1}\big)\widehat{T}_k(E_i)E_j\varphi + E_j\widehat{T}_k(E_i)E_j.\end{aligned}$$

By a similar computation as above using (ii) and (iii) of Lemma 4.4 we have that

$$E_k\Lambda_1 = \Lambda_1E_k + \Lambda_2K_k, \tag{4.25}$$

where

$$\begin{aligned}\Lambda_2 = {}& \big(1+q^{-2}\big)\big(\varphi^2E_j - \big(q+q^{-1}\big)\varphi E_j\varphi + E_j\varphi^2\big)\\ &+ [2]\big\{\big[\widehat{T}_k(E_i)E_i + q^{-4}E_i\widehat{T}_k(E_i)\big]E_j + E_j\big[\widehat{T}_k(E_i)E_i + q^{-4}E_i\widehat{T}_k(E_i)\big]\big\}\\ &- [2]\big\{\widehat{T}_k(E_i)E_jE_i + q^{-4}E_iE_j\widehat{T}_k(E_i)E_j\big\}.\end{aligned}$$

Also, by a similar computation we have that

$$E_k\Lambda_3 = \Lambda_2E_k + \Lambda_3K_k, \tag{4.26}$$

where

$$\begin{aligned}\Lambda_3 = [3]!\big\{&\varphi E_iE_j + q^{-2}E_i\varphi E_j + E_j\varphi E_i\\ &+ q^{-2}E_jE_i\varphi - \big(q+q^{-1}\big)\varphi E_jE_i - \big(q+q^{-1}\big)q^{-2}E_iE_j\varphi\big\}.\end{aligned}$$

Again, by a direct computation using Lemmas 4.3 and 4.4, we have that $E_k\Lambda_3 = \Lambda_3E_k + [2](q^2+q^{-2})(E_i^2E_j - (q+q^{-1})E_iE_jE_j + E_jE_i^2)$, i.e.,

$$E_k\Lambda_3 = \Lambda_3E_k, \tag{4.27}$$

by the Serre relation (4.2).

Now, since $K_k\Lambda_1 = q^{-2}\Lambda_1K_k$, by (4.23) and (4.24) we have that

$$F_kE_k\Lambda = q^4\Lambda F_kE_k + F_k\Lambda_1K_k, \qquad E_kF_k\Lambda = q^4\Lambda E_kF_k + q^2\Lambda_1F_kK_k.$$

Subtracting the first formula from the second formula we have that

$$[2]\big(q^2+q^{-2}\big)\Lambda = F_k\Lambda_1 - q^2\Lambda_1F_k. \tag{4.28}$$

Similarly, by (4.25) and (4.28) we have that

$$[3]!\Lambda_1 = F_k\Lambda_2 - \Lambda_2F_k. \tag{4.29}$$

By (4.26) and (4.29) we have that

$$[3]!\Lambda_2 = F_k\Lambda_3 - q^{-2}\Lambda_3F_k, \tag{4.30}$$

and by (4.27) and (4.30) we have that $[3]\Lambda_3 = \Lambda_3 - q^{-2}\Lambda_3$, which means that $\Lambda_3 = 0$. Thus by (4.30) it follows that $\Lambda_2 = 0$, and by (4.29) it follows that $\Lambda_1 = 0$. So, by (4.28) we have $\Lambda = 0$. This completes the proof. □

Now we consider the most complicated case that $a_{ki} = 2, a_{kj} = -1$, $a_{ij} = -1$ to show that (4.11) holds. As in 4.6 we modify the definition of T_k as follows (cf. (2.6)). Put

$$\widehat{T}_k(E_i) = F_k^{(2)}E_i - q^{-1}F_kE_iF_k + q^{-2}E_iF_k^{(2)}, \qquad \widehat{T}_k(E_j) = -E_kE_j + q^{-1}E_jE_k,$$
$$\Delta = \big(\widehat{T}_k(E_i)\big)^2\widehat{T}_k(E_j) + \widehat{T}_k(E_j)\big(\widehat{T}_k(E_i)\big)^2 - \big(q+q^{-1}\big)\widehat{T}_k(E_i)\widehat{T}_k(E_j)\widehat{T}_k(E_i). \tag{4.31}$$

Then (4.11) is equivalent to $\Delta = 0$. We shall use Lemmas 4.3, 4.4 frequently by similar methods to Proposition 4.5 to show the following

Proposition 4.6. *Assume that $a_{ki} = 2, a_{kj} = -1$ and $a_{ij} = -1$. Then $\Delta = 0$ and therefore* (4.11) *holds.*

Proof. By Lemmas 4.3, 4.4 and a direct computation we have that

$$E_k\Delta = q\Delta E_k + q\Delta_1K_k, \tag{4.32}$$

where

$$\begin{aligned}\Delta_1 = {}&\big[\widehat{T}_k(E_i)\varphi\widehat{T}_k(E_j) - \big(q+q^{-1}\big)\widehat{T}_k(E_i)\widehat{T}_k(E_j)\varphi + \widehat{T}_k(E_j)\widehat{T}_k(E_i)\varphi\big]\\ &+ q^{-2}\big[\varphi\widehat{T}_k(E_i)\widehat{T}_k(E_j) - \big(q+q^{-1}\big)\varphi\widehat{T}_k(E_j)\widehat{T}_k(E_i) + \widehat{T}_k(E_j)\varphi\widehat{T}_k(E_i)\big],\end{aligned}$$

and φ is given by (4.21). By similar computations we have that

$$E_k\Delta_1 = q\Delta_1E_k + q\Delta_2K_k, \tag{4.33}$$

where

$$\begin{aligned}\Delta_2 = {}&\big(1+q^{-2}\big)\big[\varphi^2\widehat{T}_k(E_j) - \big(q+q^{-1}\big)\varphi\widehat{T}_k(E_j)\varphi + \widehat{T}_k(E_j)\varphi^2\big]\\ &+ [2]\big\{\widehat{T}_k(E_i)E_i\widehat{T}_k(E_j) + q^{-4}E_i\widehat{T}_k(E_i)\widehat{T}_k(E_j)\\ &+ \widehat{T}_k(E_j)\widehat{T}_k(E_i)E_i + q^{-4}\widehat{T}_k(E_j)E_i\widehat{T}_k(E_i)\\ &- \big(q+q^{-1}\big)\big[\widehat{T}_k(E_i)\widehat{T}_k(E_j)E_i + q^{-4}E_i\widehat{T}_k(E_j)\widehat{T}_k(E_i)\big]\big\},\\ E_k\Delta_2 = {}&q\Delta_2E_k + q[3]!\Delta_3K_k,\\ \Delta_3 = {}&\big[\varphi E_i\widehat{T}_k(E_j) - \big(q+q^{-1}\big)\varphi\widehat{T}_k(E_j)E_i + \widehat{T}_k(E_j)\varphi E_i\big]\\ &+ q^{-2}\big[E_i\varphi\widehat{T}_k(E_j) - \big(q+q^{-1}\big)E_i\widehat{T}_k(E_j)\varphi + \widehat{T}_k(E_j)E_i\varphi\big].\end{aligned}$$

By the Serre relation $E_i^2E_j-(q+q^{-1})E_iE_jE_i+E_jE_i^2=0$ and a direct computation we have that $\Delta_3=0$. Hence

$$E_k\Delta_2=q\Delta_2E_k. \tag{4.34}$$

Now we compute Δ in another direction as follows. At first, since $a_{ki}=2$ and $a_{kj}=-1$, by Lemmas 4.2 and 4.4, we have that

$$F_k\Delta=q^4\Delta F_k-q^4\Delta_1'K_k^{-1}, \tag{4.35}$$

where

$$\Delta_1'=\big(\widehat{T}_k(E_i)\big)^2E_j-\big(q+q^{-1}\big)\widehat{T}_k(E_i)E_j\widehat{T}_k(E_i)+E_j\big(\widehat{T}_k(E_i)\big)^2.$$

By Lemma 4.4 and a direct computation it follows that

$$E_k\Delta_1'=q^{-1}\Delta_1'E_k-\Delta+q^{-1}\Delta_2'K_k, \tag{4.36}$$

where

$$\begin{aligned}\Delta_2'={}&\widehat{T}_k(E_i)\varphi E_j+E_j\widehat{T}_k(E_i)\varphi-\big(q+q^{-1}\big)\widehat{T}_k(E_i)E_j\varphi\\&+q^{-2}\big[\varphi\widehat{T}_k(E_i)E_j+E_j\varphi\widehat{T}_k(E_i)-\big(q+q^{-1}\big)\varphi E_j\widehat{T}_k(E_i)\big].\end{aligned}$$

By the Serre relation $E_i^2E_j-(q+q^{-1})E_iE_jE_i+E_jE_i^2=0$ and a direct computation it follows that

$$-E_k\Delta_2'+q^{-1}\Delta_2'E_k=\Delta_1. \tag{4.37}$$

Furthermore, by a direct computation we have that

$$F_k\Delta_2'-q^2\Delta_2'F_k=[2]\big(q^2+q^{-2}\big)\Delta_1'. \tag{4.38}$$

By (4.32) and (4.35), we have that

$$\begin{aligned}F_kE_k\Delta&=q^5\Delta F_kE_k-q^3\Delta_1'E_kK_k^{-1}+qF_k\Delta_1K_k,\\E_kF_k\Delta&=q^5\Delta E_kF_k-q^4E_k\Delta_1'K_k^{-1}+q^3\Delta_1\Delta_1K_k.\end{aligned}$$

It follows that

$$\begin{aligned}Q_k\Delta&=q^5\Delta Q_k-q^4\big(E_k\Delta_1'-q^{-1}\Delta_1'E_k\big)K_k^{-1}+q\big(q^2\Delta_1F_k-F_k\Delta_1\big)K_k\\&=q^5\Delta Q_k-q^4\big(-\Delta+q^{-1}\Delta_2'K_k\big)K_k^{-1}+q\big(q^2\Delta_1F_k-F_k\Delta_1\big)K_k,\end{aligned}$$

where in the last equality we use (4.36). So we have that

$$[2]\left(q^2+q^{-2}\right)\Delta K_k=-q^2\Delta_2'+\left(q^2\Delta_1 F_k-F_k\Delta_1\right)K_k. \tag{4.39}$$

Similarly, by (4.32) and (4.39) we have that

$$\Delta_2 F_k-F_k\Delta_2=\left(q^{-3}+q^3\right)\Delta_1. \tag{4.40}$$

By this with (4.34) and (4.33) we have

$$Q_k\Delta_2=q\Delta_2 Q_k-\left(q^{-3}+q^3\right)(E_k\Delta_1-q\Delta_1 E_k)=q\Delta_2 Q_k-\left(q^{-3}+q^3\right)q\Delta_2 K_k.$$

It follows that $\Delta_2=0$ and, by (4.40), $\Delta_1=0$. Also, by (4.39),

$$[2]\left(q^2+q^{-2}\right)\Delta K_k=-q^2\Delta_2'. \tag{4.41}$$

From this and (4.37) we have that $E_k\Delta_2'=q^{-1}\Delta_2'E_k$. By this and (4.38), (4.36), (4.41) it follows that

$$\begin{aligned}Q_k\Delta_2'-q\Delta_2'Q_k&=-[2]\left(q^2+q^{-2}\right)\left(-\Delta+q^{-1}\Delta_2'K_k\right)\\&=q^2\Delta_2'K_k^{-1}+[2]\left(q^2+q^{-2}\right)q^{-1}\Delta_2'K_k,\end{aligned}$$

from which we have that

$$\left\{\frac{q^{-3}-q}{q-q^{-1}}-[2]\left(q^{-2}+q^2\right)q^{-1}\right\}\Delta_2'=\frac{q^{-3}-q-q^3+q^{-5}}{q-q^{-1}}\Delta_2'=0,$$

and $\Delta_2'=0$. By (4.41) it follows that $\Delta=0$. This completes the proof. □

Note that the case that $a_{ki}=0, a_{kj}=2$ (respectively $a_{ki}=-1, a_{kj}=2$) is similar to Proposition 4.5 (respectively Proposition 4.6). So we have shown that for any $1\leqslant k\leqslant \ell+2$, T_k respects the Serre relation (4.2).

5. The GIM Serre relations

In this section we assume that $a_{ij}=2$ but $i\neq j$ and show that for each $1\leqslant k\leqslant \ell+2$, T_k respects the defining relation (Re. 3). By applying the involution (2.4) of U_q, it suffices to show that

$$\sum_{s=0}^{s=3}(-1)^s\left(T_k(E_i)\right)^{(3-s)}T_k(F_j)\left(T_k(E_i)\right)^{(s)}=0, \tag{5.1}$$

$$\left[T_k(E_i),T_k(E_j)\right]=0. \tag{5.2}$$

We show (5.1) holds first, which is much more complicated than (5.2). If either $k=i$ or $k=j$ then (5.1) follows by the following Propositions 5.1 and 5.2, respectively. If $k\neq i$ and

$k \neq j$, since $a_{ij} = 2$ and $i \neq j$, without loss of generality, we may assume that $i = \ell + 1$ and $j = \ell + 2$, i.e., we may assume that $\alpha_1 = \theta - \delta_1$ and $\alpha_2 = \theta - \delta_2$. Hence for any $1 \leqslant k \leqslant \ell + 2$ we have that

$$a_{k1} = (\theta - \delta_1)(h_k) = \theta(h_k) = (\theta - \delta_2)(h_k) = a_{jk}.$$

It follows that there are exactly following two cases needed to be considered:

(i) $a_{ki} = a_{kj} = 0$;
(ii) $a_{ki} = a_{kj} = -1$.

For the first case that $a_{ki} = a_{kj} = 0$, it is easy to see that (5.1) holds. For the second case we show (5.2) in Proposition 5.3. At the end of this section we show (5.2) in Proposition 5.4.

Proposition 5.1. *Assume that $a_{ij} = 2$ and $i \neq j$. Then it holds that*

$$\sum_{s=0}^{s=3} (-1)^s \big(T_i(E_i)\big)^{(3-s)} T_i(F_j) \big(T_i(E_i)\big)^{(s)} = 0. \tag{5.3}$$

Proof. Keep notation in Section 3. In this case $F_i F_j = F_j F_i$ and by (2.7) we have that

$$T_i(F_j) = q^{-2}\big(F_j E_i^{(2)} - q E_i F_j E_i + q^2 E_i^{(2)} F_j\big).$$

Since $T_i(E_i) = -F_i K_i$, we have that

$$\big(T_i(E_i)\big)^{(2)} = q^6 K_{2i} F_i^{(2)}, \qquad \big(T_i(E_i)\big)^{(3)} = -q^{12} K_{3i} F_i^{(3)}.$$

Also, by Lemma 3.3, we have that

$$E_i F_i^{(3)} = F_i^{(3)} E_i + \frac{1}{[3]}\big(F_i^{(2)} Q_i + F_i x_i F_i\big), \tag{5.4}$$

where x_i is given by (4.31), and

$$E_i^{(2)} F_i^{(3)} = F_i^{(3)} E_i^{(2)} + z_i,$$

where

$$z_i = \frac{1}{[3]}\left\{F_i^{(2)} E_i \bar{x}_i + F_i^2 E_i \frac{q^2 K_i - q^2 K_i^{-1}}{q - q^{-1}} + F_i Q_i \frac{q^2 K_i - q^2 K_i^{-1}}{q - q^{-1}} + P_i F_i\right\},$$

and $\bar{x}_i$ is given by (3.8), P_i is given by (3.9). Using Lemma 3.3 and above two formulae with a lengthy but straightforward computation to move forward the letter F_i in each summand of the left-hand side of (5.3), we have that

$$\text{L.H.S. of (5.3)} = -q^{10}K_{3i}F_i^{(3)}\bigl(T_i(F_j)-T_i(F_j)\bigr)+q^{-2}\sum_{t=1}^{9}V_t, \tag{5.5}$$

where

$$\begin{aligned}
V_1 &= q^{10}K_{3i}F_i^{(2)}F_jE_i\bar{x}_i,\\
V_2 &= -q^8K_{3i}F_iF_j(F_iE_iQ_i+P_i) = -q^8K_{3i}F_i^2F_jE_iQ_i - q^{-8}K_{3i}F_iF_jP_i,\\
V_3 &= \frac{1}{[3]}q^6K_{3i}F_j\biggl[F_i^{(2)}E_i\bar{x}_i+F_i^2E_i\frac{q^2K_i-q^2K_i^{-1}}{q-q^{-1}}+F_iQ_i\frac{q^2K_i-q^2K_i^{-1}}{q-q^{-1}}+P_iF_i\biggr],\\
V_4 &= -q^{11}K_{3i}F_i^{(2)}Q_iF_jE_i - q^{11}K_{3i}F_i^{(2)}E_iF_jQ_i,\\
V_5 &= q^9K_{3i}F_i^2x_iF_jE_i+q^9K_{3i}F_iE_iF_jF_ix_i\\
&= q^9K_{3i}F_i^2x_iF_jE_i+q^9K_{3i}F_i^2E_iF_jx_i+q^9K_{3i}F_iQ_iF_jx_i,\\
V_6 &= -\frac{1}{[3]}q^7K_{3i}\bigl[\bigl(F_i^{(2)}Q_i+F_ix_iF_i\bigr)F_jE_i+E_iF_j\bigl(F_i^{(2)}Q_i+F_ix_iF_i\bigr)\bigr]\\
&= -\frac{1}{[3]}q^7K_{3i}F_i^{(2)}Q_iF_jE_i-\frac{1}{[3]}q^7K_{3i}F_ix_iF_iF_jE_i-\frac{1}{[3]}K_{3i}F_i^{(2)}E_iF_jQ_i\\
&\quad -\frac{1}{[3]}q^7K_{3i}F_ix_iF_jQ_i-\frac{[2]}{[3]}q^7K_{3i}F_i^{(2)}E_ix_iF_j-\frac{[2]}{[3]}q^7K_{3i}F_ix_i^2F_j,\\
V_8 &= -q^{10}K_{3i}F_i^2E_iQ_iF_j-q^{10}K_{3i}F_iP_iF_j,\\
V_9 &= \frac{1}{[3]}q^8K_{3i}F_i^{(2)}E_i\bar{x}_iF_j+\frac{1}{[3]}q^8K_{3i}F_i^2E_i\frac{q^{-2}K_i-q^2K_i^{-1}}{q-q^{-1}}F_j\\
&\quad +\frac{1}{[3]}q^8K_{3i}F_iQ_i\frac{q^{-2}K_i-q^2K_i^{-1}}{q-q^{-1}}F_j+\frac{1}{[3]}K_{3i}P_iF_iF_j,
\end{aligned}$$

where x_i is given by (3.7), $\bar{x}_i$ is given by (3.8) and P_j is given by (3.9).

By V_1, V_2, V_3, V_4 and V_6 it follows that the right coefficient of $K_{3i}F_i^{(2)}F_jE_i$ in $\sum_{t=1}^{9}V_t$ is given by $q^6(\lambda_1+\lambda_2)$, where

$$\begin{aligned}
\lambda_1 &= q^4\bar{x}_i+q^3[2]x_i+\frac{[2]}{[3]}\frac{q^{-2}K_i-q^2K_i^{-1}}{q-q^{-1}}\\
&\quad -q\frac{[2]}{[3]}\frac{q^{-2}(1+q^{-2})K_i-q^2(1+q^2)K_i^{-1}}{q^2-q^{-2}}+q^3[2]x_i\\
&= \frac{1}{[3]}\frac{t_1(q)K_i-t_1(q)K_i^{-1}}{q^2-q^{-2}},
\end{aligned}$$

and $t_1(q)=q^8+3q^6+5q^4+6q^2+q^{-2}+4$, while

$$\lambda_2 = -\left(q^2[2] + q^5 + \frac{1}{[3]}q\right)Q_i = -\frac{1}{[3]}\frac{t_1(q)K_i - t_1(q)K_i^{-1}}{q^2 - q^{-2}},$$

so $\lambda_1 + \lambda_2 = 0$ and therefore the summand $K_{3i}F_i^{(2)}F_jE_i$ in $\sum_{t=1}^9 V_t$ vanishes.

By V_4, V_5, V_6, V_7, V_8 and V_9 it follows that the right coefficient of $K_{3i}F_i^{(2)}E_iF_j$ in $\sum_{t=1}^9 V_t$ is given by $q^7(\sigma_1 + \sigma_2)$, where

$$\begin{aligned}\sigma_1 &= q^2[2]x_i - \left\{\frac{[2]}{[3]} - q^5 - q\frac{1}{[3]}\right\}\frac{q^{-2}(1+q^{-2})K_i - q^2(1+q^2)K_i^{-1}}{q^2 - q^{-2}} \\ &\quad - q^3[2]\frac{q^{-2}K_i - q^2K_i}{q - q^{-1}} + q\frac{[2]}{[3]}\frac{q^{-4}K_i - q^4K_i^{-1}}{q - q^{-1}} \\ &= \frac{1}{[3]}\frac{t_2(q)K_i - t_2(q)K_i^{-1}}{q^2 - q^{-2}},\end{aligned}$$

and $t_2(q) = q^7 + 2q^5 + 2q^3 + 2q + q^{-1}$, while

$$\sigma_2 = -\left(q^4 + \frac{1}{[3]}\right)Q_i = -\frac{1}{[3]}\frac{t_2(q)K_i - t_2(q)K_i^{-1}}{q^2 - q^{-2}},$$

so $\sigma_1 + \sigma_2 = 0$ and therefore the summand $K_{3i}F_i^{(2)}E_iF_j$ in $\sum_{t=1}^9 V_t$ vanishes.

So it remains to check that the summand $K_{3i}F_iF_j$ in $\sum_{t=1}^9 V_t$ vanishes. Note that $K_iF_j = q^{-2}F_jK_i$ since $a_{ij} = 2$. By V_2, V_3, V_5, V_6, V_8 and V_9, it follows that the right coefficient of $K_{3i}F_iF_j$ in $\sum_{t=1}^9 V_t$ is given by $q^6\sum_{t=1}^9 W_t$, where

$$\begin{aligned}W_1 &= -q^2P_i = -q^2\left\{\frac{1}{[2]^2}\frac{(1+q^2)K_{2i} + (1+q^2)K_{2i}^{-1}}{(q - q^{-1})^2} - \frac{1}{(q - q^{-1})^2}\right\}, \\ W_2 &= \frac{1}{[3]}Q_i\frac{q^{-2}K_i - q^2K_i^{-1}}{q - q^{-1}} = \frac{q^{-2}K_{2i} + q^2K_{2i}^{-1}}{[3](q - q^{-1})^2} - \frac{q^{-2} + q^2}{[3](q - q^{-1})^2}, \\ W_3 &= \frac{1}{[3]}\left\{\frac{1}{[2]^2}\frac{q^{-4}(q^{-2}+1)K_{2i} + q^4(1+q^2)K_{2i}^{-1}}{(q - q^{-1})^2} - \frac{1}{(q - q^{-1})^2}\right\}, \\ W_4 &= q^3\frac{(q^{-2}K_i - q^2K_i^{-1})((1+q^{-2})K_i - (1+q^2)K_i^{-1})}{(q - q^{-1})(q^2 - q^{-2})} \\ &= q^3\left\{\frac{q^{-2}(1+q^{-2})K_{2i} + q^2(1+q^2)K_{2i}^{-1}}{(q - q^{-1})(q^2 - q^{-2})} - \frac{q^2 + q^{-2} + 2}{(q - q^{-1})(q^2 - q^{-2})}\right\}, \\ W_5 &= -q\frac{[2]}{[3]}\frac{(q^{-2}(1+q^{-2})K_i - (q^2(1+q^2)K_i^{-1})^2}{(q - q^{-1})^2} \\ &= -q\frac{[2]}{[3]}\left\{\frac{q^{-4}(1+q^{-2})^2K_{2i} + (q^4(1+q^2)^2K_{2i}^{-1}}{(q - q^{-1})^2} - 2\frac{1 - q^{-4}}{(q - q^{-1})^2}\right\},\end{aligned}$$

$$\begin{aligned}
W_6 &= -q\frac{1}{[3]}\frac{(q^{-2}(1+q^{-2})K_i - q^2(1+q^2)K_i)(K_i - K_i^{-1})}{(q^2-q^{-2})(q-q^{-1})}\\
&= -q\frac{1}{[3]}\left\{\frac{q^{-2}(1+q^{-2})K_{2i} + q^2(1+q^2)K_{2i}}{(q^2-q^{-2})(q-q^{-1})} - \frac{q^{-2}+q^{-4}+q^2+q^4}{(q^2-q^{-2})(q-q^{-1})}\right\},\\
W_7 &= -q^4\left\{\frac{q^{-4}(q^{-2}+1)K_{2i} + q^4(1+q^2)K_{2i}^{-1}}{(q^2-q^{-2})^2} - \frac{1}{(q-q^{-1})^2}\right\},\\
W_8 &= \frac{q^2}{[3]}\frac{(q^{-2}K_i - q^2K_i^{-1})(q^{-4}K_i - q^4K_i)}{(q-q^{-1})^2}\\
&= \frac{q^2}{[3]}\left\{\frac{q^{-6}K_{2i} + q^6K_{2i}^{-1}}{(q-q^{-1})^2} - \frac{q^2+q^{-2}}{(q-q^{-1})^2}\right\},\\
W_9 &= \frac{q^2}{[3]}\left\{\frac{q^{-8}(q^{-2}+1)K_{2i} + q^8(1+q^2)K_{2i}^{-1}}{(q^2-q^{-2})^2} - \frac{1}{(q-q^{-1})^2}\right\}.
\end{aligned}$$

Put $\sum_{t=1}^{9} W_t = \zeta_1(q)K_{2i} + \zeta_2(q)K_{2i}^{-1} + \zeta_3(q)$. From W_t $(1 \leqslant t \leqslant 9)$ as above and by a direct computation we have that

$$\zeta_1(q) = \zeta_2(q) = \zeta_3(q) = 0,$$

which means that $\sum_{t=1}^{9} W_t = 0$ and therefore the summand $K_{3i}F_iF_j$ in $\sum_{t=1}^{9} V_t$ vanishes.

So far we have shown that $\sum_{t=1}^{9} V_t = 0$. By (5.5) it follows that (5.3) holds. This completes the proof. □

Proposition 5.2. *Assume that $a_{ij} = 2$ and $i \neq j$. Then it holds that*

$$\sum_{s=0}^{s=3}(-1)^s\big(T_j(E_i)\big)^{(3-s)}T_j(F_j)\big(T_j(E_i)\big)^{(s)} = 0. \tag{5.6}$$

As in Section 4 we put $\widehat{T}_j(F_j) = T_j(F_j)$ and

$$\widehat{T}_j(E_i) = q^{-2}T_j(E_i) = F_j^{(2)}E_i - q^{-1}F_jE_iF_j + q^{-2}E_iF_j^{(2)}.$$

Then by the definition of T_j (cf. (2.6)), the equality (5.6) is equivalent to

$$\sum_{s=0}^{s=3}(-1)^s\big(\widehat{T}_j(E_i)\big)^{(3-s)}\widehat{T}_j(F_j)\big(\widehat{T}_j(E_i)\big)^{(s)} = 0. \tag{5.7}$$

Put

$$\varphi_{ij} = F_jE_i - q^{-2}E_iF_j. \tag{5.8}$$

Let us recall Lemmas 4.3 and 4.4. For convenience, we collect the following facts which will be used in the proof of this proposition as follows.

Lemma 5.1. *Assume that $a_{ij}=2$ but $i\neq j$. Then*

(i) $K_j\varphi_{ij}=\varphi_{ij}K_j$, $K_j\widehat{T}_j(E_i)=q^{-2}\widehat{T}_j(E_i)K_j$;
(ii) $\varphi_{ij}E_i^2-(1+q^2)E_i\varphi_{ij}E_i+q^2E_i^2\varphi_{ij}=0$;
(iii) $E_j\widehat{T}_j(E_i)=\widehat{T}_j(E_i)E_j+\varphi_{ij}K_j$, $F_j\widehat{T}_j(E_i)=q^2\widehat{T}_j(E_j)F_j$;
(iv) $E_j\varphi_{ij}=\varphi_{ij}E_j+[2]E_iK_j$;
(v) $\widehat{T}_j(E_i)=\frac{1}{[2]}\{F_j\varphi_{ij}-\varphi_{ij}F_j\}$;
(vi) $\widehat{T}_j(E_i)\widehat{T}_j(F_j)=q^{-2}\widehat{T}_j(F_j)\widehat{T}_j(E_i)-q^{-2}\varphi_{ij}$.

In particular, (ii) is the Serre relation (2.2). By (vi) it follows that

$$\text{L.H.S. of (5.7)}=-\frac{q^2}{[3]}\left\{\left(\widehat{T}_j(E_i)\right)^2\varphi_{ij}-\left(1+q^2\right)\widehat{T}_j(E_i)\varphi_{ij}\widehat{T}_j(E_i)+q^2\varphi_{ij}\left(\widehat{T}_j(E_i)\right)^2\right\}.$$

Put

$$\Omega=\left(\widehat{T}_j(E_i)\right)^2\varphi_{ij}-\left(1+q^2\right)\widehat{T}_j(E_i)\varphi_{ij}\widehat{T}_j(E_i)+q^2\varphi_{ij}\left(\widehat{T}_j(E_i)\right)^2. \tag{5.9}$$

To prove Proposition 5.2, it suffices to show that $\Omega=0$. We have that

$$K_j\Omega=q^{-4}\Omega K_j,\qquad F_j\Omega=q^4\Omega F_j. \tag{5.10}$$

Assume that $a_{ij}=2$ and $i\neq j$. By using Lemma 5.1 and straightforward computations we have the following formulae

$$E_j\Omega=\Omega E_j+\Omega_1K_j,\qquad K_j\Omega_1=q^{-2}\Omega_1K_j, \tag{5.11}$$

where

$$\begin{aligned}\Omega_1={}&-q^2\widehat{T}_j(E_i)\varphi_{ij}^2+\left(q^2+q^{-2}\right)\widehat{T}_j(E_i)\varphi_{ij}\widehat{T}_j(E_i)-q^{-2}\varphi_{ij}^2\widehat{T}_j(E_i)\\&+[2]\left\{\left(\widehat{T}_j(E_i)\right)^2E_i-\left(1+q^{-2}\right)\widehat{T}_j(E_i)E_i\widehat{T}_j(E_i)+q^{-2}E_i\left(\widehat{T}_j(E_i)\right)^2\right\},\end{aligned}$$

$$E_j\Omega_1=\Omega_1E_j+[3]!\Omega_2K_j,\qquad K_j\Omega_2=\Omega_2K_j, \tag{5.12}$$

where

$$\Omega_2=\left(\varphi_{ij}\widehat{T}_j(E_i)E_i-\widehat{T}_j(E_i)E_i\varphi_{ij}\right)+q^2\left(E_i\widehat{T}_j(E_i)\varphi_{ij}-\varphi_{ij}E_i\widehat{T}_j(E_i)\right),$$

$$E_j\Omega_2=\Omega_2E_j+\Omega_3K_j,\qquad K_j\Omega_3=q^2\Omega_3K_j, \tag{5.13}$$

where

$$\begin{aligned}\Omega_3 = {} & q^2\varphi_{ij}^2 E_i - \left(q^2+q^{-2}\right)\varphi_{ij}E_i\varphi_{ij} + q^{-2}E_i\varphi_{ij}^2 \\ & - [2]\left\{\widehat{T}_j(E_i)E_i^2 - \left(1+q^{-2}\right)E_i\widehat{T}_j(E_i)E_i + q^{-2}E_i^2\widehat{T}_j(E_i)\right\}.\end{aligned}$$

Also, by using Lemma 5.1 and straightforward computations we have that

$$E_j\Omega_3 = \Omega_3 E_j - [2]\left(q^2+q^{-2}\right)\left\{\varphi_{ij}E_i^2 - \left(1+q^2\right)E_i\varphi_{ij}E_i + q^2E_i^2\varphi_{ij}\right\}K_j.$$

By (ii) of Lemma 5.1, it follows that

$$E_j\Omega_3 = \Omega_3 E_j. \tag{5.14}$$

Proof of Proposition 5.2. By (5.10), (5.11) we have that

$$E_jF_j\Omega = q^4\Omega E_jF_j + q^2\Omega_1F_jK_j, \qquad F_jE_j\Omega = q^4\Omega F_jE_j + F_j\Omega_1K_j,$$

so $Q_j\Omega = q^4\Omega Q_j + (q^2\Omega_1F_j - F_j\Omega_1)K_j$. Since $K_j\Omega = q^{-4}\Omega K_j$, we have that

$$[2]\left(q^2+q^{-2}\right)\Omega = F_j\Omega_1 - q^2\Omega_1F_j. \tag{5.15}$$

Similarly, by (5.12) and (5.15) we have that

$$\Omega_1 = F_j\Omega_2 - \Omega_2F_j. \tag{5.16}$$

By (5.13) and (5.16) we have that

$$[3]!\Omega_2 = F_j\Omega_3 - q^{-2}\Omega_3F_j. \tag{5.17}$$

By (5.14) and (5.17) we have that $([2]-[3]!)\Omega_3 = 0$, which means that $\Omega_3 = 0$. By (5.17) it follows that $\Omega_2 = 0$. By (5.16) it follows that $\Omega_1 = 0$. At last, by (5.15) it follows that $\Omega = 0$ as required. This completes the proof. □

Proposition 5.3. *Assume that $a_{ki} = a_{kj} = -1$, $a_{ij} = 2$ and $i \neq j$. Then the equality* (5.1) *holds.*

The proof is highly combinatorial. In this case (5.1) is equivalent to:

$$\begin{aligned}&\left(T_k(E_i)\right)^3T_k(F_j) - [3]\left(T_k(E_i)\right)^2T_k(F_j)T_k(E_i) \\ &\quad + [3]\left(T_k(E_i)\right)T_k(F_j)\left(T_k(E_i)\right)^2 - T_k(F_j)\left(T_k(E_i)\right)^3 = 0.\end{aligned} \tag{5.18}$$

The aim of following computations is to establish (5.39) below. At first, since $a_{ki} = -1$, by Serre relation (2.1) we have that

$$\begin{aligned}\big(T_k(E_i)\big)^2 &= E_k^{(2)}E_i^2 - \big(1+q^{-2}\big)E_kE_i^{(2)}E_k + q^{-2}E_i^2E_k\\ &= E_k^{(2)}E_i^2 - \big(1+q^{-2}\big)E_iE_k^{(2)}E_i + q^{-2}E_i^2E_k.\end{aligned} \tag{5.19}$$

In particular we have that

$$E_kE_i^{(2)}E_k = E_iE_k^{(2)}E_i. \tag{5.20}$$

We need the following

Lemma 5.2. *Assume that $a_{ki}=-1$. Then*

$$\big(T_k(E_i)\big)^3 = -E_k^{(3)}E_i^3 + q^{-1}E_k^{(2)}E_i^3E_k - q^{-2}E_kE_i^3E_k^{(2)} + q^{-3}E_i^3E_k^{(3)}. \tag{5.21}$$

Proof. By (5.19) we have that

$$\begin{aligned}\big(T_k(E_i)\big)^3 = {}&-E_kE_iE_k^{(2)}E_i^2 + \big(1+q^{-2}\big)E_kE_iE_kE_i^{(2)}E_k - q^{-2}E_kE_i^3E_k^{(2)}\\ &+ q^{-1}E_iE_kE_k^{(2)}E_i^2 - q^{-1}\big(1+q^{-2}\big)E_iE_k^2E_i^{(2)}E_k\\ &+ q^{-3}E_iE_kE_i^2E_k^{(2)}.\end{aligned}$$

By Serre relation (2.1) we have that

$$E_k^2E_iE_kE_i^2 = [2]^2E_k^2E_iE_kE_i^2 - [2]E_k^3E_i^3 - E_k^2E_i^3E_k.$$

Since $[2]^2-1=[3]$, it follows that $E_k^2E_iE_kE_i^2 = \frac{[2]}{[3]}E_k^3E_i^3 + \frac{1}{[3]}E_k^2E_i^3E_k$. So

$$-E_kE_iE_k^{(2)}E_i^2 = -E_k^2E_iE_kE_i^2 + \frac{1}{[2]}E_k^3E_i^3 = -E_k^{(3)}E_i^3 - \frac{[2]}{[3]}E_k^{(2)}E_i^3E_k \tag{5.22}$$

by noting that $-\frac{[2]}{[3]}+\frac{1}{[2]} = -\frac{1}{[3]!}$. Also, by (5.20) it follows that

$$\begin{aligned}E_iE_k^{(2)}E_i^{(2)}E_k &= \frac{1}{[2]}E_iE_k^{(2)}E_iE_iE_k = \frac{1}{[2]}E_kE_i^{(2)}E_kE_iE_k\\ &= \frac{1}{[2]}E_kE_i^{(2)}E_k^{(2)}E_i + \frac{1}{[2]}E_kE_i^{(2)}E_iE_k^{(2)}\\ &= \frac{1}{[2]^2}E_iE_k^{(2)}E_i^{(2)}E_k + \frac{1}{[2]^2}E_k^{(2)}E_iE_i^{(2)}E_k + \frac{1}{[2]^2}E_kE_i^3E_k^{(2)},\end{aligned}$$

which means that

$$\left(1-\frac{1}{[2]^2}\right)E_iE_k^{(2)}E_i^{(2)}E_k = \frac{1}{[2]^2}E_kE_i^3E_k^{(2)} + \frac{1}{[2]^3}E_k^{(2)}E_i^3E_k,$$

i.e.,

$$E_i E_k^{(2)} E_i^{(2)} E_k = \frac{1}{[3]} E_k E_i^3 E_k^{(2)} + \frac{1}{[3]!} E_k^{(2)} E_i^3 E_k$$

by noting that $1 - \frac{1}{[2]^2} = \frac{[3]}{[2]^2}$. So we have that

$$\begin{aligned}(1+q^{-2}) E_k E_i E_k E_i^{(2)} E_k &= (1+q^{-2})\{E_k^{(2)} E_i E_i^{(2)} E_k + E_i E_k^{(2)} E_i^{(2)} E_k\} \\ &= \frac{(1+q^{-2})[2]}{[3]} E_k^{(2)} E_i^3 E_k + \frac{1+q^{-2}}{[3]} E_k E_i^3 E_k^{(2)} \end{aligned} \tag{5.23}$$

by noting that $\frac{1}{[2]} + \frac{1}{[3]!} = \frac{[2]}{[3]}$. Furthermore,

$$\begin{aligned}E_i E_k E_k^{(2)} E_i^2 &= E_i E_k^{(2)} E_k E_i^2 = E_k E_i E_k^2 - E_k^{(2)} E_i E_k E_i^2 \\ &= \frac{[3]}{[2]}\left\{\frac{[2]}{[3]} E_k^3 E_i^3 + \frac{1}{[3]} E_k^2 E_i^3 E_k\right\} - E_k^3 E_i^3 = \frac{1}{[2]} E_k^2 E_i^3 E_k,\end{aligned}$$

i.e.,

$$q^{-1} E_i E_k E_k^{(2)} E_i^2 = q^{-1} E_k^{(2)} E_i^3 E_k. \tag{5.24}$$

We continue to compute other summands in $(T_k(E_i))^3$,

$$\begin{aligned}&-q^{-1}(1+q^{-2}) E_i E_k^2 E_i^{(2)} E_k \\ &\quad = -q^{-1}(1+q^{-2})[2] E_i E_k^{(2)} E_i^{(2)} E_k \\ &\quad = -(1+q^{-2})^2 \left\{\frac{1}{[3]} E_k E_i^3 E_k^{(2)} + \frac{1}{[3]!} E_k^{(2)} E_i^3 E_k\right\} \\ &\quad = -\frac{(1+q^{-2})^2}{[3]} E_k E_i^3 E_k^{(2)} - \frac{(1+q^{-2})^2}{[3]!} E_k^{(2)} E_i^3 E_k. \end{aligned} \tag{5.25}$$

Also we have that

$$\begin{aligned}E_i E_k E_i^2 E_k^{(2)} &= \frac{1}{[2]} E_i E_k E_i^2 E_k E_k = \frac{1}{[2]} E_i E_i E_k^2 E_i E_k = E_i^2 E_k^{(2)} E_i E_k \\ &= [2]^2 E_i E_k E_i^2 E_k^{(2)} - [2] E_k E_i^3 E_k^{(2)} - [3] E_i^3 E_k^{(3)}.\end{aligned}$$

It follows that

$$q^{-3} E_i E_k E_i^2 E_k^{(2)} = \frac{[2]}{[3]} q^{-3} E_k E_i^3 E_k^{(2)} + q^{-3} E_i^3 E_k^{(3)}. \tag{5.26}$$

By (5.22), (5.23), (5.24), (5.25) and (5.26) we have that

$$\begin{aligned}\big(T_k(E_i)\big)^3 = {} & -E_k^{(3)}E_i^3 + q^{-3}E_i^3E_k^{(3)} \\ & + \left\{-\frac{[2]}{[3]} + \frac{[2](1+q^{-2})}{[3]} + q^{-1} - \frac{(1+q^{-2})^2}{[3]!}\right\}E_k^{(2)}E_i^3E_k \\ & + \left\{\frac{1+q^{-2}}{[3]} - q^{-2} - \frac{(1+q^{-2})^2}{[3]} + \frac{[2]}{[3]}q^{-3}\right\}E_kE_i^3E_k^{(2)}.\end{aligned}$$

But

$$\begin{aligned}-\frac{[2]}{[3]} + \frac{[2](1+q^{-2})}{[3]} + q^{-1} - \frac{(1+q^{-2})^2}{[3]!} &= q^{-1}, \\ \frac{1+q^{-2}}{[3]} - q^{-2} - \frac{(1+q^{-2})^2}{[3]} + \frac{[2]}{[3]}q^{-3} &= -q^{-2},\end{aligned}$$

and the lemmas follows. □

Lemma 5.3. *Assume that $a_{ki} = a_{kj} = -1$ and $i \neq j$. Then*

$$\big(T_k(E_i)\big)^3 T_k(F_j) = qF_kA_{11} - A_{11}F_k + A_{12}K_k^{-1}, \tag{5.27}$$

where

$$\begin{aligned}A_{11} &= -E_k^{(3)}E_i^3F_j + q^{-1}E_k^{(2)}E_i^3F_jE_k - q^{-2}E_kE_i^3F_jE_k^{(2)} + q^{-3}E_i^3F_jE_k^{(3)}, \\ A_{12} &= E_k^{(2)}E_i^3F_j - q^{-2}E_kE_i^3F_jE_k + q^{-4}E_i^3F_jE_k^{(2)}.\end{aligned}$$

Proof. Note that $T_k(F_j) = -F_jF_k + qF_kF_j$. Since $a_{kj} = -1$, we have that $F_jE_k = E_kF_j$. By (5.21) it follows that $-(T_k(E_i))^3F_jF_k = -A_{11}F_k$. So it remains to compute $(T_k(E_i))^3F_kF_j$.

At first, by Lemma 3.3, it follows that

$$E_k^{(3)}F_k = F_kE_k^{(3)} + E_k^{(2)}y_k, \qquad y_k = \frac{q^2K_k - q^{-2}K_k^{-1}}{q-q^{-1}}.$$

Therefore, by this formula and Lemma 3.3 it follows that

$$\begin{aligned}-E_k^{(3)}E_i^3F_kF_j &= -E_k^{(3)}F_kE_i^3F_j = -\big(F_kE_k^{(3)} + E_k^{(2)}y_k\big)E_i^3F_j \\ &= -F_kE_k^{(3)}E_i^3F_j - E_k^{(2)}E_i^3F_jQ_k, \\ q^{-1}E_k^{(2)}E_i^3E_kF_kF_j &= q^{-1}E_k^{(2)}E_i^3F_kE_kF_j + q^{-1}E_k^{(2)}E_i^3F_j\frac{qK_k - q^{-1}K_k^{-1}}{q-q^{-1}} \\ &= q^{-1}F_kE_k^{(2)}E_i^3F_jE_k + q^{-1}E_kE_i^3F_jE_k\bar{x}_k\end{aligned}$$

$$
\begin{aligned}
&\quad + E_k^{(2)} E_i^3 F_j \frac{K_k - q^{-2} K_k^{-1}}{q - q^{-1}}, \\
-q^{-2} E_k E_i^3 E_k^{(2)} F_k F_j &= -q^{-2} E_k E_i^3 \big(F_k E_k^{(2)} + E_k \bar{x}_k\big) F_j \\
&= -q^{-2} F_k E_k E_i^3 F_j E_k^{(2)} - E_i^3 F_j E_k^{(2)} \frac{K_k - q^{-4} K_k^{-1}}{q - q^{-1}} \\
&\quad - E_k E_i^3 F_j E_k \frac{q^{-1}(q^2+1) K_k - (q^{-2}+1) q^{-3} K_k^{-1}}{q^2 - q^{-2}}, \\
q^{-3} E_i^3 E_k^{(3)} F_k F_j &= q^{-3} E_i^3 \big(F_k E_k^{(3)} + E_k^{(2)} y_k\big) F_j \\
&= q^{-3} F_k E_i^3 F_j E_k^3 + E_i^3 F_j E_k^{(2)} \frac{K_k - q^{-6} K_k^{-1}}{q - q^{-1}}.
\end{aligned}
$$

Summing these formulae up we have that

$$
\begin{aligned}
\big(T_k(E_i)\big)^3 F_k F_j &= F_k A_{11} + E_k^{(2)} E_i^3 F_j \left\{ \frac{K_k - q^{-2} K_k^{-1}}{q - q^{-1}} - Q_k \right\} \\
&\quad + E_k E_i^3 F_j E_k \left\{ q^{-1} \bar{x}_k - \frac{(q^{-1}+q) K_k - (q^{-5}+q^{-3}) K_k^{-1}}{q^2 - q^{-2}} \right\} \\
&\quad + E_i^3 F_j E_k^{(2)} \left\{ \frac{K_k - q^{-6} K_k^{-1}}{q - q^{-1}} - \frac{K_k - q^{-4} K_k^{-4}}{q - q^{-1}} \right\} \\
&= F_k A_{11} + \big\{ q^{-1} E_k^{(2)} E_i^3 F_j - q^{-3} E_k E_i^3 F_j E_k + q^{-5} E_i^3 F_j E_k^{(2)} \big\} K_k^{-1}
\end{aligned}
$$

and the lemma follows. □

In a completely similar way we have the following

Lemma 5.4. *Assume that $a_{ki} = a_{kj} = -1$ and $i \neq j$. Then*

$$
-T_k(F_j)\big(T_k(E_i)\big)^3 = q F_k A_{41} - A_{41} F_k + A_{42} K_k^{-1}, \tag{5.28}
$$

where

$$
\begin{aligned}
A_{41} &= E_k^{(3)} F_j E_i^3 - q^{-1} E_k^{(2)} F_j E_i^3 E_k + q^{-2} E_k F_j E_i^3 E_k^{(2)} - q^{-3} F_j E_i^3 E_k^{(3)}, \\
A_{42} &= -E_k^{(2)} F_j E_i^3 + q^{-2} E_k F_j E_i^3 E_k - q^{-4} F_j E_i^3 E_k^{(2)}.
\end{aligned}
$$

We continue to compute other two summands of (5.18). At first, for $a_{ki} = a_{kj} = -1$ and $i \neq j$, by Serre relation (2.1) we have that

$$
T_k(E_i) F_j T_k(E_i) = E_k^{(2)} E_i F_j E_i - q^{-1} E_k E_i F_j E_i E_k + q^{-2} E_i F_j E_i E_k^{(2)}. \tag{5.29}
$$

Lemma 5.5. *Assume that $a_{ki} = a_{kj} = -1$ and $i \neq j$. Then*

$$\begin{aligned}\big(T_k(E_i)\big)^2 F_j T_k(E_i) = {} & -E_k^{(3)} E_i^2 F_j E_i + q^{-1} E_k^{(2)} E_i^2 F_j E_i E_k \\ & - q^{-2} E_k E_i^2 F_j E_i E_k^{(2)} + q^{-3} E_i^2 F_j E_i E_k^{(3)}. \end{aligned} \tag{5.30}$$

Proof. By (5.29) we have that $(T_k(E_i))^2 F_j T_k(E_i) = \sum_{i=1}^6 R_i$, where

$$\begin{aligned} R_1 &= -E_k E_i E_k^{(2)} E_i F_j E_i, & R_2 &= q^{-1} E_k E_i E_k E_i F_j E_i E_k, \\ R_3 &= -q^{-2} E_k E_i^2 F_j E_i E_k^{(2)}, & R_4 &= q^{-1} E_i E_k E_k^{(2)} E_i F_j E_i, \\ R_5 &= -q^{-2} E_i E_k^2 E_i F_j E_i E_k, & R_6 &= q^{-3} E_i E_k E_i F_j E_i E_k^{(2)}. \end{aligned}$$

Note that

$$\begin{aligned} -E_k E_i E_k^{(2)} E_i F_j E_i &= -E_k^2 E_i E_k E_i F_j E_i + E_k E_k^{(2)} E_i^2 F_j E_i \\ &= -[2]^2 E_k E_i E_k^{(2)} E_i F_j E_i + E_i E_k^3 E_i F_j E_i + E_k E_k^{(2)} E_i^2 F_j E_i. \end{aligned}$$

So $([2]^2 - 1) E_k E_i E_k^{(2)} E_i F_j E_i = E_i E_k^3 E_i F_j E_i + E_k E_k^{(2)} E_i^2 F_j E_i$ and

$$R_1 = -\frac{1}{[3]} E_i E_k^3 E_i F_j E_i - E_k^{(3)} E_i^2 F_j E_i.$$

Since, again by Serre relation (2.1),

$$\begin{aligned} R_2 &= q^{-1} E_k^{(2)} E_i^2 F_j E_i E_k + q^{-1} E_i E_k^{(2)} E_i F_j E_i E_k, \\ R_5 &= -q^{-2}\big(q + q^{-1}\big) E_i E_k^{(2)} E_i F_j E_i E_k, \end{aligned}$$

which means that $R_2 + R_5 = q^{-1} E_k^{(2)} E_i^2 F_j E_i E_k - q^{-3} E_i E_k^{(2)} E_i F_j E_i E_k$. But

$$\begin{aligned} E_i E_k^{(2)} E_i F_j E_i E_k &= E_i E_k E_i E_k F_j E_i E_k - E_i^2 E_k^{(2)} F_j E_i E_k \\ &= E_i E_k E_i F_j E_k^{(2)} E_i + E_i E_k E_i F_j E_i E_k^{(2)} - E_i^2 E_k^{(2)} F_j E_i E_k, \end{aligned}$$

It follows that

$$R_2 + R_5 + R_6 = q^{-1} E_k^{(2)} E_i^2 F_j E_i E_k - q^{-3} E_i E_k E_i F_j E_k^{(2)} E_i + q^{-3} E_i^2 E_k^{(2)} F_j E_i E_k.$$

Also, by Serre relation (2.1) and (5.20) it follows that

$$\begin{aligned}
E_iE_kE_iF_jE_k^{(2)}E_i &= E_iE_kE_iE_k^{(2)}F_jE_i\\
&= E_iE_k^2E_iE_kF_jE_i - E_iE_kE_k^{(2)}E_iF_jE_i\\
&= E_kE_i^2E_k^2F_jE_i - \frac{1}{[2]}E_iE_k^3E_iF_jE_i\\
&= [2]^2E_iE_kE_iE_k^{(2)}F_jE_i - E_i^2E_k^3F_jE_i - \frac{1}{[2]}E_iE_k^3E_iF_jE_i,\\
E_i^2E_k^3F_jE_i &= E_i^2F_jE_kE_k^2E_i\\
&= [2]E_i^2F_jE_k^2E_iE_k - E_i^2F_jE_kE_iE_k^2\\
&= [2]^2E_i^2F_jE_kE_iE_k^2 - [2]E_i^2F_jE_iE_k^3 - E_i^2F_jE_kE_iE_k^2\\
&= [3]E_i^2F_jE_kE_iE_k^2 - [2]E_i^2F_jE_iE_k^3\\
&= [3]E_i^2F_jE_k^{(2)}E_iE_k + [3]E_i^2F_jE_iE_k^{(2)}E_k - [2]E_i^2F_jE_iE_k^3\\
&= [3]E_i^2E_k^{(2)}F_jE_iE_k - E_i^2F_jE_iE_k^{(2)}E_k
\end{aligned}$$

by noting that $[3] = [2]^2 - 1$. So we have that

$$\begin{aligned}
E_iE_kE_iF_jE_k^{(2)}E_i = {} & [2]^2E_iE_kE_iE_k^{(2)}F_jE_i - [3]E_i^2E_k^{(2)}F_jE_iE_k\\
& + E_i^2F_jE_iE_k^{(2)}E_k - \frac{1}{[2]}E_iE_k^3E_iF_jE_i,
\end{aligned}$$

which means that

$$E_iE_kE_iF_jE_k^{(2)}E_i = E_i^2E_k^{(2)}F_jE_iE_k - E_i^2F_jE_iE_k^{(3)} + E_iE_k^{(3)}E_iF_jE_i,$$

and hence

$$R_2 + R_5 + R_6 = q^{-1}E_k^{(2)}E_i^2F_jE_iE_k + q^{-3}E_i^2F_jE_iE_k^{(3)} - q^{-3}E_iE_k^{(3)}E_iF_jE_i.$$

So we have that

$$\begin{aligned}
\big(T_k(E_i)\big)^2F_jT_k(E_i) &= (R_1 + R_3 + R_4) + (R_2 + R_5 + R_6)\\
&= -E_k^{(3)}E_i^2F_jE_i + q^{-1}E_k^{(2)}E_i^2F_jE_iE_k - q^{-2}E_kE_i^2F_jE_iE_k^{(2)}\\
&\quad + q^{-3}E_i^2F_jE_iE_k^{(3)} + \left\{-\frac{1}{[3]} + \frac{1}{[2]}q^{-1} - \frac{1}{[3]!}q^{-3}\right\}E_iE_k^3E_iF_jE_i
\end{aligned}$$

and the lemma follows by noting that $-\frac{1}{[3]} + \frac{1}{[2]}q^{-1} - \frac{1}{[3]!}q^{-3} = 0$. □

Similarly we have the following

Lemma 5.6. *Assume that $a_{ki} = a_{kj} = -1$ and $i \neq j$. Then*

$$\begin{aligned}\big(T_k(E_i)\big) F_j T_k(E_i)^2 = &-E_k^{(3)} E_i F_j E_i^2 + q^{-1} E_k^{(2)} E_i F_j E_i^2 E_k \\ &- q^{-2} E_k E_i F_j E_i^2 E_k^{(2)} + q^{-3} E_i F_j E_i^2 E_k^{(3)}. \end{aligned} \tag{5.31}$$

Proof. By (5.29) we have that $(T_k(E_i))^2 F_j T_k(E_i) = \sum_{i=1}^{6} S_i$, where

$$\begin{aligned} S_1 &= -E_k^{(2)} E_i F_j E_i E_k E_i, & S_2 &= q^{-1} E_k^{(2)} E_i F_j E_i^2 E_k, \\ S_3 &= q^{-1} E_k E_i F_j E_i E_k^2 E_i, & S_4 &= -q^{-2} E_k E_i F_j E_i E_k E_i E_k, \\ S_5 &= -q^{-2} E_i F_j E_i E_k^{(2)} E_k E_i, & S_6 &= q^{-3} E_i F_j E_i E_k^{(2)} E_i E_k. \end{aligned}$$

By using Serre relation (2.1) and (5.20) we have that

$$\begin{aligned} S_1 &= -E_k^{(3)} E_i F_j E_i^2 + \frac{1}{[3]!} E_i F_j E_i E_k^3 E_i - E_k^{(2)} E_i F_j E_i^{(2)} E_k - E_i E_k^{(2)} F_j E_i^{(2)} E_k, \\ S_3 &= \big(1 + q^{-2}\big) E_k^{(2)} E_i F_j E_i^{(2)} E_k + \big(1 + q^{-2}\big) E_i E_k^{(2)} F_j E_i^{(2)} E_k, \\ S_4 &= -q^{-2} E_k^{(2)} E_i F_j E_i^{(2)} E_k - q^{-2} E_i E_k^{(2)} F_j E_i^{(2)} E_k - q^{-2} E_k E_i F_j E_i^2 E_k^{(2)}, \\ S_6 &= q^{-3} E_i F_j E_i^2 E_k^{(3)} + \frac{q^{-3}}{[3]} E_i F_j E_i E_k^3 E_i. \end{aligned}$$

It follows that

$$\begin{aligned} \big(T_k(E_i)\big) F_j T_k(E_i)^2 &= (S_1 + S_3 + S_4) + (S_2 + S_5 + S_6) \\ &= -E_k^{(3)} E_i F_j E_i^2 + q^{-1} E_k^{(2)} E_i F_j E_i^2 E_k - q^{-2} E_k E_i F_j E_i^2 E_k^{(2)} \\ &\quad + q^{-3} E_i F_j E_i^2 E_k^{(3)} + \left\{ -\frac{1}{[3]!} + \frac{1}{[2]} q^{-2} + \frac{1}{[3]} q^{-3} \right\} E_i F_j E_i E_k^3 E_i \end{aligned}$$

and the lemma follows by noting that $\frac{1}{[3]!} - \frac{1}{[2]} q^{-2} + \frac{1}{[3]} q^{-3} = 0$. $\square$

Lemma 5.7. *Assume that $a_{ki} = a_{kj} = -1$ and $i \neq j$. Then*

$$-[3]\big(T_k(E_i)\big)^2 T_k(F_j) T_k(E_i) = q F_k A_{21} - A_{21} F_k + A_{22} K_k^{-1}, \tag{5.32}$$

where

$$\begin{aligned} A_{21} = {} & [3] E_k^{(3)} E_i^2 F_j E_i - [3] q^{-1} E_k^{(2)} E_i^2 F_j E_i E_k \\ & + [3] q^{-2} E_k E_i^2 F_j E_i E_k^{(2)} - [3] q^{-3} E_i^2 F_j E_i E_k^{(3)}, \\ A_{22} = {} & -[3] E_k^{(2)} E_i^2 F_j E_i + [3] q^{-2} E_k E_i^2 F_j E_i E_k - [3] q^{-4} E_i^2 F_j E_i E_k^{(2)}. \end{aligned}$$

Proof. Since $a_{kj}=-1$, we have that $F_jE_k=E_kF_j$. By (iii) of Lemma 4.2 and definition (2.7) of $T_k(F_j)$, we have that

$$\begin{aligned}
&\bigl(T_k(E_i)\bigr)^2T_k(F_j)T_k(E_i)\\
&\quad=-\bigl(T_k(E_i)\bigr)^2F_jT_k(E_i)F_k+qF_k\bigl(T_k(E_i)\bigr)^2F_jT_k(E_i)\\
&\qquad+\bigl\{\bigl(T_k(E_i)\bigr)^2F_jE_i+\bigl(1+q^{-2}\bigr)q^{-1}T_k(E_i)E_iF_jT_k(E_i)\bigr\}K_k^{-1}. \qquad (5.33)
\end{aligned}$$

By (5.19) we have that

$$\begin{aligned}
\bigl(T_k(E_i)\bigr)^2F_jE_i&=E_k^{(2)}E_i^2F_jE_i-\bigl(1+q^{-2}\bigr)E_kE_i^{(2)}E_kF_jE_i+q^{-2}E_i^2E_k^{(2)}F_jE_i\\
&=E_k^{(2)}E_i^2F_jE_i-\bigl(1+q^{-2}\bigr)E_iE_kE_iE_kF_jE_i\\
&\quad+\bigl(1+2q^{-2}\bigr)E_i^2E_k^{(2)}F_jE_i. \qquad (5.34)
\end{aligned}$$

On the other hand, by (ii) of Lemma 4.2 it follows that

$$\begin{aligned}
&\bigl(1+q^{-2}\bigr)q^{-1}T_k(E_i)E_iF_jT_k(E_i)\\
&\quad=\bigl(1+q^{-2}\bigr)E_iT_k(E_i)F_jT_k(E_i)\\
&\quad=\bigl(1+q^{-2}\bigr)E_iE_kE_iF_jE_kE_i-\bigl(1+q^{-2}\bigr)q^{-1}E_iE_kE_iF_jE_iE_k\\
&\qquad-\bigl(1+q^{-2}\bigr)q^{-1}E_i^2E_kF_jE_kE_i+\bigl(1+q^{-2}\bigr)q^{-2}E_i^2E_kF_jE_iE_k.
\end{aligned}$$

Note that

$$\begin{aligned}
&-\bigl(1+q^{-2}\bigr)q^{-1}E_iE_kE_iF_jE_iE_k\\
&\quad=-\bigl(1+q^{-2}\bigr)q^{-1}E_i^{(2)}E_kF_jE_iE_k-\bigl(1+q^{-2}\bigr)q^{-1}E_kE_i^{(2)}F_jE_iE_k\\
&\quad=-q^{-2}E_i^2F_jE_kE_iE_k-q^{-2}E_kE_i^2F_jE_iE_k\\
&\quad=-q^{-2}E_i^2F_jE_k^{(2)}E_i-q^{-2}E_i^2F_jE_iE_k^{(2)}-q^{-2}E_kE_i^2F_jE_iE_k\\
&\qquad-\bigl(1+q^{-2}\bigr)q^{-1}E_i^2E_kF_jE_kE_i\\
&\quad=-\bigl(1+q^{-2}\bigr)^2E_i^2E_k^{(2)}F_jE_i,
\end{aligned}$$

and

$$\begin{aligned}
&\bigl(1+q^{-2}\bigr)q^{-2}E_i^2E_kF_jE_iE_k\\
&\quad=\bigl(1+q^{-2}\bigr)q^{-2}E_i^2F_jE_kE_iE_k\\
&\quad=\bigl(1+q^{-2}\bigr)q^{-2}E_i^2F_jE_k^{(2)}E_i+\bigl(1+q^{-2}\bigr)q^{-2}E_i^2F_jE_iE_k^{(2)}.
\end{aligned}$$

So we have that

$$\begin{aligned}
&\big(1+q^{-2}\big)q^{-1}T_k(E_i)E_iF_jT_k(E_i)\\
&\quad=\big(1+q^{-2}\big)E_iE_kE_iF_jE_kE_i-q^{-2}E_kE_i^2F_jE_iE_k+q^{-4}E_i^2F_jE_iE_k^{(2)}\\
&\qquad-\big\{q^{-2}+\big(1+q^{-2}\big)^2-\big(1+q^{-2}\big)q^{-2}\big\}E_i^2F_jE_k^{(2)}E_i.
\end{aligned}\tag{5.35}$$

By (5.34) and (5.35) we have that

$$\begin{aligned}
&\big(T_k(E_i)\big)^2F_jE_i+\big(1+q^{-2}\big)q^{-1}T_k(E_i)E_iF_jT_k(E_i)\\
&\quad=E_k^{(2)}E_i^2F_jE_i-q^{-2}E_kE_i^2F_jE_i+q^{-4}E_i^2F_jE_iE_k^{(2)}\\
&\qquad+\big\{1+2q^{-2}-q^{-2}-\big(1+q^{-2}\big)^2+\big(1+q^{-2}\big)q^{-2}\big\}E_i^2E_k^{(2)}F_jE_i\\
&\quad=E_k^{(2)}E_i^2F_jE_i-q^{-2}E_kE_i^2F_jE_i+q^{-4}E_i^2F_jE_iE_k^{(2)},
\end{aligned}$$

by noting that $1+2q^{-2}-q^{-2}-(1+q^{-2})^2+(1+q^{-2})q^{-2}=0$. By this formula and (5.30), (5.33) the lemma follows. □

Similarly we have the following

Lemma 5.8. *Assume that $a_{ki}=a_{kj}=-1$ and $i\neq j$. Then*

$$[3]T_k(E_i)T_k(F_j)\big(T_k(E_i)\big)^2=qF_kA_{31}-A_{31}F_k+A_{32}K_k^{-1},\tag{5.36}$$

where

$$\begin{aligned}
A_{31}&=-[3]E_k^{(3)}E_iF_jE_i^2+[3]q^{-1}E_k^{(2)}E_iF_jE_i^2E_k\\
&\qquad-[3]q^{-2}E_kE_iF_jE_i^2E_k^{(2)}+[3]q^{-3}E_iF_jE_i^2E_k^{(3)},\\
A_{32}&=[3]E_k^{(2)}E_iF_jE_i^2-[3]q^{-2}E_kE_iF_jE_i^2E_k+[3]q^{-4}E_iF_jE_i^2E_k^{(2)}.
\end{aligned}$$

Proof. Since $a_{kj}=-1$, we have that $F_jE_k=E_kF_j$. By (iii) of Lemma 4.2 and definition (2.7) of $T_k(F_j)$, we have that

$$\begin{aligned}
&T_k(E_i)T_k(F_j)\big(T_k(E_i)\big)^2\\
&\quad=-T_k(E_i)F_j\big(T_k(E_i)\big)^2+qF_kT_k(E_i)F_j\big(T_k(E_i)\big)^2\\
&\qquad+\big\{\big(1+q^{-2}\big)T_k(E_i)F_jT_k(E_i)E_i+q^{-2}E_iF_j\big(T_k(E_i)\big)^2\big\}K_k^{-1}.
\end{aligned}\tag{5.37}$$

Set $(1+q^{-2})T_k(E_i)F_jT_k(E_i)E_i=T_1+T_2+T_3+T_4$, where

$$\begin{aligned}
T_1 &= \bigl(1+q^{-2}\bigr)E_kE_iE_kF_jE_i^2 = \bigl(1+q^{-2}\bigr)E_k^{(2)}E_iF_jE_i^2 + \bigl(1+q^{-2}\bigr)E_iE_k^{(2)}F_jE_i^2,\\
T_2 &= -\bigl(1+q^{-2}\bigr)q^{-1}E_kE_iF_jE_iE_kE_i\\
&= -\bigl(1+q^{-2}\bigr)q^{-1}E_kE_iF_jE_i^{(2)}E_k - \bigl(1+q^{-2}\bigr)q^{-1}E_kE_iF_jE_kE_i^{(2)}\\
&= -q^{-2}E_kE_iF_jE_i^2E_k - q^{-2}E_k^{(2)}E_iF_jE_i^2 - q^{-2}E_iE_k^{(2)}F_jE_i^2,\\
T_3 &= -\bigl(1+q^{-2}\bigr)q^{-1}E_iE_kF_jE_kE_i^2 = -\bigl(1+q^{-2}\bigr)^2E_iE_k^{(2)}F_jE_i^2,\\
T_4 &= \bigl(1+q^{-2}\bigr)q^{-2}E_iE_kF_jE_iE_kE_i.
\end{aligned}$$

By (5.19) and setting $q^{-2}E_iF_j(T_k(E_i))^2 = T_5+T_6+T_7$, we have that

$$\begin{aligned}
T_5 &= q^{-2}E_iF_jE_k^{(2)}E_i^2 = q^{-2}E_iE_k^{(2)}F_jE_i^2,\\
T_6 &= -\bigl(1+q^{-2}\bigr)q^{-2}E_iF_jE_kE_i^{(2)}E_k\\
&= -\bigl(1+q^{-2}\bigr)q^{-2}E_iE_kF_jE_iE_kE_i + \bigl(1+q^{-2}\bigr)q^{-2}E_iE_k^{(2)}F_jE_i^2,\\
T_7 &= q^{-4}E_iF_jE_i^2E_k^{(2)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\bigl(1+q^{-2}\bigr)T_k(E_i)F_jT_k(E_i)E_i + q^{-2}E_iF_j\bigl(T_k(E_i)\bigr)^2\\
&\quad= \sum_{i=1}^{7}T_i = E_k^{(2)}E_iF_jE_i^2 - q^{-2}E_kE_iF_jE_i^2E_k + q^{-4}E_iF_jE_i^2E_k^{(2)}\\
&\qquad+ \bigl\{1+q^{-2}+q^{-2}-q^{-2}-\bigl(1+q^{-2}\bigr)^2+\bigl(1+q^{-2}\bigr)q^{-2}\bigr\}E_iE_k^{(2)}F_jE_i^2\\
&\quad= E_k^{(2)}E_iF_jE_i^2 - q^{-2}E_kE_iF_jE_i^2E_k + q^{-4}E_iF_jE_i^2E_k^{(2)}.
\end{aligned}\tag{5.38}$$

By (5.31), (5.37) and (5.38) the lemma follows. □

Proof of Proposition 5.3. Set

$$S = E_i^3F_j - [3]E_i^2F_jE_i + [3]E_iF_jE_i^2 - F_jE_i^2 = 0,$$

which is one of the Serre relations (2.2). Then by (5.27), (5.28), (5.32) and (5.36) we have that

$$\begin{aligned}
&\text{L.H.S. of (5.18)}\\
&\quad= qF_k\bigl\{-E_k^3S + q^{-1}E_k^{(2)}SE_k - q^{-2}E_kSE_k^{(2)} + q^{-3}SE_k^{(3)}\bigr\}\\
&\qquad- \bigl\{-E_k^{(3)}S + q^{-1}E_k^{(2)}SE_k - q^{-2}E_kSE_k^{(2)} + q^{-3}SE_k^{(3)}\bigr\}F_k\\
&\qquad+ \bigl\{E_k^{(2)}S - q^{-2}E_kSE_k + q^{-4}SE_k^{(2)}\bigr\}K_k^{-1} = 0,
\end{aligned}\tag{5.39}$$

which completes the proof. □

Now we check that T_k $(1 \leqslant k \leqslant \ell+2)$ respects the Serre relation (2.3).

Proposition 5.4. *Assume that $a_{ij}=2$ and $i \neq j$. Then, for any $1 \leqslant k \leqslant \ell+2$, it holds that $[T_k(E_i), T_k(E_j)] = [T_k(F_i), T_k(F_j)] = 0$.*

Proof. It suffices to check that $[T_k(E_i), T_k(E_j)] = 0$.

(i) Assume that $k=i$. Then we have that $T_i(E_i) = -F_iK_i$,

$$T_i(E_j) = q^2\bigl\{F_i^{(2)}E_j - q^{-1}F_iE_jF_i + q^{-2}E_jF_i^{(2)}\bigr\}, \qquad K_iT_i(E_j) = q^{-2}T_i(E_j)K_i.$$

By Serre relation (2.2) it follows that

$$\begin{aligned}
&\bigl[T_i(E_i), T_i(E_j)\bigr] \\
&\quad = T_i(E_i)T_i(E_j) - T_i(E_j)T_i(E_i) \\
&\quad = -\frac{1}{[2]}\bigl\{F_i^3E_j - [3]F_i^2E_jF_i + [3]F_iE_jF_i^2 - E_jF_i^3\bigr\}K_i = 0.
\end{aligned}$$

(ii) Assume that $k=j$. Then we have that $T_j(E_j) = -F_jK_j$,

$$T_j(E_i) = q^2\bigl\{F_j^{(2)}E_i - q^{-1}F_jE_iF_j + q^{-2}E_iF_j^{(2)}\bigr\}, \qquad K_jT_j(E_i) = q^{-2}T_j(E_i)K_j.$$

It follows that

$$\begin{aligned}
\bigl[T_j(E_i), T_j(E_j)\bigr] &= T_j(E_i)T_j(E_i) - T_j(E_j)T_j(E_i) \\
&= -\frac{1}{[2]}\bigl\{F_j^3E_i - [3]F_j^2E_iF_j + [3]F_jE_iF_j^2 - E_iF_j^3\bigr\}K_j = 0
\end{aligned}$$

by Serre relation (2.2).

(iii) Assume that $k \neq i$ and $k \neq j$. As before, there are only two cases needed to be checked.

(a) $a_{ki} = a_{kj} = 0$. This is clear since $[T_k(E_i), T_k(E_j)] = [E_i, E_j] = 0$.

(b) $a_{ki} = a_{kj} = -1$. By $E_iE_j = E_jE_i$ and Serre relation (2.1) we have that

$$\begin{aligned}
&\bigl[T_k(E_i), T_k(E_j)\bigr] \\
&\quad = T_k(E_i)T_k(E_i) - T_k(E_j)T_k(E_i) \\
&\quad = E_kE_iE_kE_j - q^{-1}E_kE_iE_jE_k - q^{-1}E_iE_k^2E_j + q^{-2}E_iE_kE_jE_k \\
&\qquad - E_kE_jE_kE_i + q^{-1}E_kE_jE_iE_k + q^{-1}E_jE_k^2E_i - q^{-2}E_jE_kE_iE_k \\
&\quad = E_kE_iE_kE_j + q^{-1}E_iE_jE_k^2 - \bigl(1+q^{-2}\bigr)E_iE_kE_jE_k + q^{-2}E_iE_kE_jE_k \\
&\qquad - E_kE_jE_kE_i + \bigl(1+q^{-2}\bigr)E_jE_kE_iE_k - q^{-1}E_jE_iE_k^2 - q^{-2}E_jE_kE_iE_k \\
&\quad = E_kE_iE_kE_j - E_iE_kE_jE_k - E_kE_jE_kE_i + E_jE_kE_iE_k
\end{aligned}$$

$$= \frac{1}{[2]}\big\{E_k^{(2)}E_iE_j + E_iE_k^{(2)}E_j - E_iE_jE_k^{(2)}$$
$$- E_k^{(2)}E_jE_i - E_jE_k^{(2)}E_i + E_jE_k^{(2)}E_i + E_jE_iE_k^{(2)}\big\} = 0.$$

This completes the proof. □

6. The inverse of Lusztig symmetries

By a completely way we can show that for each $1 \leqslant i \leqslant \ell + 2$, T_i' given by (2.8) and (2.9) is an endomorphism of the quantized GIM algebra U_q. In this section we shall prove the remaining part of Theorem 2.1, i.e., T_i' is the inverse of T_i. As we shall see, the most complicated computation arise in the case that $a_{ij} = 2$, $i \neq j$.

Proposition 6.1. *For each $1 \leqslant i \leqslant \ell + 2$, it holds that $T_iT_i' = T_i'T_i = 1$ as endomorphisms of U_q.*

We prove this result by checking it on generators. It is easy to see that $T_iT_i'(K_\beta) = T_i'T_i(K_\beta) = K_\beta$, $T_iT_i'(D_j^\pm) = T_i'T_i(D_j^\pm) = D_j^\pm$ for $j = 1, 2$ and $\beta \in \Gamma$. Moreover, we have that

$$T_iT_i'(E_i) = T_i\big(-K_i^{-1}F_i\big) = -T_i\big(K_i^{-1}\big)T_i(F_i) = -K_i\big(-K_i^{-1}E_i\big) = E_i,$$
$$T_i'T_i(E_i) = T_i'(-F_iK_i) = -T_i'(F_i)T_i(K_i) = -(-E_iK_i)K_i^{-1} = E_i.$$

Similarly, $T_iT_i'(F_i) = F_i = T_i'T_i(F_i)$. So, by the involution (2.4) of U_q it remains to show that

$$T_i'T_i(E_j) = T_iT_i'(E_j) = E_j \quad \text{for } i \neq j. \tag{6.1}$$

The case that $a_{ij} = 0$ is clear. So it suffices to consider the case $a_{ij} = -1$ (see Lemma 6.1 below) and the case that $a_{ij} = 2$ (see Lemmas 6.2 and 6.3 below).

Lemma 6.1. *Assume that $a_{ij} = -1$. Then the equality* (6.1) *holds.*

Proof. Since T_i and T_i' are endomorphisms, by the definition (2.6) of T_i and the definition (2.8) of T_i', we have that

$$T_i'T_i(E_j) = T_i'\big(-E_iE_j + q^{-1}E_jE_i\big) = -T_i'(E_i)T_i'(E_j) + q^{-1}T_i'(E_j)T_i'(E_i)$$
$$= -K_i^{-1}\big\{E_j(F_iE_i - E_iF_i) + q^{-1}(E_iF_i - F_iE_i)E_j\big\}$$
$$= -K_i^{-1}\big\{-E_jQ_i + q^{-1}Q_iE_j\big\} = -K_i^{-1}E_j\frac{-1 + q^{-2}}{q - q^{-1}}K_i = E_j.$$

Similarly we have that $T_iT_i'(E_j) = E_j$. This completes the proof. □

Lemma 6.2. *Assume that $a_{ij}=2$ and $i\neq j$. Then $T_i'T_i(E_j)=E_j$.*

Proof. By definition, $(T_i'(F_i))^2=q^2E_i^2K_{2i}$, $K_iT_i'(E_j)=q^{-2}T_i'(E_j)K_i$. So $q^{-2}T_i'(E_j)\times(T_i'(F_i))^2=T_i'(E_j)E_i^2K_{2i}$ and

$$\big(T_i'(F_i)\big)^2T_i'(E_j)=q^{-2}E_i^2T_i'(E_j)K_{2i},\qquad T_i'(F_i)T_i'(E_j)T_i'(F_i)=E_iT_i'(E_j)E_iK_{2i}.$$

It follows that

$$\begin{aligned}
T_i'T_i(E_j)&=\frac{q^4}{[2]^2}\big\{q^{-2}E_i^2\big(E_jF_i^2-\big(1+q^{-2}\big)F_iE_jE_i+q^{-2}F_i^2E_j\big)\\
&\quad-\big(1+q^{-2}\big)E_i\big(E_jF_i^2-\big(1+q^{-2}\big)F_iE_jE_i+q^{-2}F_i^2E_j\big)E_i\\
&\quad+\big(E_jF_i^2-\big(1+q^{-2}\big)F_iE_jE_i+q^{-2}F_i^2E_j\big)E_i^2\big\}K_{2i}\\
&=\frac{q^4}{[2]^2}\left(\sum_{t=1}^{9}\xi_t\right)K_{2i},
\end{aligned}\tag{6.2}$$

where

$$\begin{aligned}
\xi_1&=q^{-2}E_i^2E_jF_i^2=q^{-2}E_jE_i^2F_i^2,\\
\xi_2&=-q^{-2}\big(1+q^{-2}\big)E_i^2F_iE_jF_i,\\
\xi_3&=q^{-4}E_i^2F_i^2E_j,\\
\xi_4&=-\big(1+q^{-2}\big)E_iE_jF_i^2E_i=-\big(1+q^{-2}\big)E_jE_iF_i^2E_i\\
&=-\big(1+q^{-2}\big)E_jE_i^2F_i^2+[2]\big(1+q^{-2}\big)E_jE_iF_ix_i\quad\text{(by Lemma 3.3)},\\
\xi_5&=\big(1+q^{-2}\big)^2E_iF_iE_jF_iE_i\\
&=\big(1+q^{-2}\big)^2E_iF_iE_jE_iF_i-\big(1+q^{-2}\big)^2E_iF_iE_jQ_i\\
&=\big(1+q^{-2}\big)^2E_iF_iE_iE_jF_i-\big(1+q^{-2}\big)^2E_iF_iE_jQ_i\\
&=\big(1+q^{-2}\big)^2E_i^2F_iE_jF_i-\big(1+q^{-2}\big)^2E_jE_iF_iQ_i-\big(1+q^{-2}\big)^2E_iF_iE_jQ_i,\\
\xi_6&=-q^{-2}\big(1+q^{-2}\big)E_iF_i^2E_jE_i=-q^{-2}\big(1+q^{-2}\big)E_iF_i^2E_iE_j\\
&=-q^{-2}\big(1+q^{-2}\big)E_i^2F_i^2E_j+[2]q^{-2}\big(1+q^{-2}\big)E_iF_iE_j\bar{x}_i,\\
\xi_7&=E_jF_i^2E_i^2=E_jE_i^2F_i^2-[2]^2E_jF_iE_iQ_i-[2]^2E_jP_i\quad\text{(by Lemma 3.3)},\\
\xi_8&=-\big(1+q^{-2}\big)F_iE_jF_iE_i^2\\
&=-\big(1+q^{-2}\big)F_iE_jE_i^2F_i+[2]\big(1+q^{-2}\big)F_iE_jE_i\bar{x}_i\\
&=-\big(1+q^{-2}\big)F_iE_i^2E_jF_i+[2]\big(1+q^{-2}\big)F_iE_jE_i\bar{x}_i
\end{aligned}$$

$$
\begin{aligned}
&= -\left(1+q^{-2}\right)E_i^2F_iE_jF_i + [2]\left(1+q^{-2}\right)E_i\bar{x}_iE_jF_i \\
&\quad + [2]\left(1+q^{-2}\right)F_iE_jE_i\bar{x}_i \\
&= -\left(1+q^{-2}\right)E_i^2F_iE_jF_i + [2]\left(1+q^{-2}\right)E_iE_jF_i\bar{x}_i \\
&\quad + [2]\left(1+q^{-2}\right)F_iE_jE_i\bar{x}_i, \\
\xi_9 &= q^{-2}F_i^2E_jE_i^2 = q^{-2}F_i^2E_i^2E_j \\
&= q^{-2}E_i^2F_i^2E_j - [2]^2q^{-2}F_iE_iQ_iE_j - q^{-2}[2]^2P_iE_j \\
&= q^{-2}E_i^2F_i^2E_j - [2]^2q^{-2}F_iE_iE_j\frac{q^2K_i - q^{-2}K_i^{-1}}{q-q^{-1}} - q^{-2}[2]^2P_iE_j.
\end{aligned}
$$

It follows that

$$
\xi_1+\xi_4+\xi_7 = [2]\left(1+q^{-2}\right)E_jE_iF_ix_i - [2]^2E_jF_iE_iQ_i - [2]^2E_jP_j, \tag{6.3}
$$

$$
\begin{aligned}
\xi_2+\xi_5+\xi_8 &= -\left(1+q^{-2}\right)^2E_jE_iF_iQ_i - \left(1+q^{-2}\right)^2E_iF_iE_jQ_i \\
&\quad + [2]\left(1+q^{-2}\right)E_jE_iF_i\bar{x}_i + [2]\left(1+q^{-2}\right)F_iE_iE_j\bar{x}_i,
\end{aligned} \tag{6.4}
$$

$$
\begin{aligned}
\xi_3+\xi_6+\xi_9 &= [2]q^{-2}\left(1+q^{-2}\right)E_iF_iE_j\bar{x}_i - [2]^2q^{-2}P_iE_j \\
&\quad - [2]^2q^{-2}F_iE_iE_j\frac{q^2K_i - q^{-2}K_i}{q-q^{-1}}.
\end{aligned} \tag{6.5}
$$

Note that

$$
\begin{aligned}
&[2]\left(1+q^{-2}\right)E_jE_iF_ix_i + [2]\left(1+q^{-2}\right)E_jE_iF_i\bar{x}_i - \left(1+q^{-2}\right)^2E_jE_iF_iQ_i \\
&\quad = \left(1+q^{-2}\right)E_jE_iF_i\left\{[2]x_i + [2]\bar{x}_i - \left(1+q^{-2}\right)Q_i\right\} = [2]^2E_jE_iF_iQ_i, \\
&[2]F_iE_iE_j\left(1+q^{-2}\right)\bar{x}_i - [2]^2q^{-2}F_iE_iE_j\frac{q^2K_i - q^{-2}K_i^{-1}}{q-q^{-1}} \\
&\quad = [2]F_iE_iE_j\left\{\frac{(1+q^{-2})((1+q^2)K_i - (1+q^{-2})K_i^{-1})}{q^2-q^{-2}} - [2]\frac{K_i - q^{-4}K_i^{-1}}{q-q^{-1}}\right\} \\
&\quad = -q^{-3}[2]^2F_iE_iE_jK_i^{-1},
\end{aligned}
$$

while

$$
\begin{aligned}
&[2]q^{-2}\left(1+q^{-2}\right)E_iF_iE_j\bar{x}_i - \left(1+q^{-2}\right)^2E_iF_iE_jQ_i \\
&\quad = \left(1+q^{-2}\right)E_iF_iE_j\left\{[2]q^{-2}\bar{x}_i - \left(1+q^{-2}\right)Q_i\right\} = q^{-3}[2]^2E_iF_iE_jK_i^{-1}.
\end{aligned}
$$

So, by (6.3), (6.4) and (6.5) it follows that

$$\sum_{t=1}^{9}\xi_t = [2]^2 E_j E_i F_i Q_i - [2]^2 E_j F_i E_i Q_i + [2]^2 q^{-3} E_i F_i E_j K_i^{-1}$$
$$- [2]^2 F_i E_i E_j K_i^{-1} - [2]^2 q^{-3} F_i E_i E_j K_i^{-1} - [2]^2 E_j P_i - [2]^2 q^{-2} P_i E_j$$
$$= [2]^2\left\{E_j Q_i^2 + q^{-3} Q_i E_j K_i^{-1} - E_j P_i - q^{-2} P_i E_j\right\}$$
$$= [2]^2 E_j\left\{\frac{-2}{(q-q^{-1})^2} + \frac{q^{-1}}{q-q^{-1}} + \frac{q^{-2}}{(q-q^{-1})^2} + \frac{1}{(q-q^{-1})^2}\right.$$
$$+ \left(\frac{1}{(q-q^{-1})^2} - \frac{q^{-2}+1}{[2]^2(q-q^{-1})} - \frac{1+q^2}{[2]^2(q-q^{-1})^2}\right)K_{2i}$$
$$\left. + \left(\frac{1}{(q-q^{-1})^2} - \frac{q^{-5}}{q-q^{-1}} - \frac{q^2+1}{[2]^2(q-q^{-1})^2} - \frac{q^{-4}+q^{-6}}{[2]^2(q-q^{-1})^2}\right)K_{2i}^{-1}\right\}.$$

But

$$\frac{-2}{(q-q^{-1})^2} + \frac{q^{-1}}{q-q^{-1}} + \frac{q^{-2}}{(q-q^{-1})^2} + \frac{1}{(q-q^{-1})^2} = 0,$$
$$\frac{1}{(q-q^{-1})^2} - \frac{q^{-2}+1}{[2]^2(q-q^{-1})} - \frac{1+q^2}{[2]^2(q-q^{-1})^2} = 0,$$
$$\frac{1}{(q-q^{-1})^2} - \frac{q^{-5}}{q-q^{-1}} - \frac{q^2+1}{[2]^2(q-q^{-1})^2} - \frac{q^{-4}+q^{-6}}{[2]^2(q-q^{-1})^2} = q^{-4},$$

it follows that $\sum_{t=1}^{9}\xi_t = [2]^2 q^{-4} E_j K_{2i}^{-1}$. By (6.2) the lemma follows. This completes the proof. □

Lemma 6.3. *Assume that $a_{ij} = 2$ and $i \neq j$. Then $T_i T_i'(E_j) = E_j$.*

Proof. By definition, $(T_i(E_i))^2 = q^{-6} E_i^2 K_{-2i}^{-1}$, $K_i^{-1} T_i(E_j) = q^2 T_i(E_j) K_i^{-1}$. By a similar argument as in Lemma 6.2, by using Lemma 3.3 we have

$$T_i T_i'(E_j) = \frac{q^{-2}}{[2]^2}\left(\sum_{t=1}^{9}\eta_t\right)K_{2i}^{-1}, \tag{6.6}$$

where

$$\eta_1 = F_i^2 E_j E_i^2,$$
$$\eta_2 = -\left(1+q^{-2}\right) F_i E_j F_i E_i^2,$$
$$\eta_3 = q^{-2} E_j F_i^2 E_i^2,$$
$$\eta_4 = -\left(1+q^2\right) E_i F_i^2 E_j E_i = -\left(1+q^2\right) F_i^2 E_i^2 E_j - [2]\left(1+q^2\right) F_i x_i E_i E_j$$

$$
\begin{aligned}
&= -\left(1+q^2\right) F_i^2 E_i^2 E_j \\
&\quad - [2]\left(1+q^2\right) F_i E_i E_j \frac{(q^4+q^2)K_i - (q^{-4}+q^{-2})K_i^{-1})}{q^2-q^{-2}}, \\
\eta_5 &= \left(1+q^{-2}\right)\left(1+q^2\right) E_i F_i E_j F_i E_i = [2]^2 F_i E_i E_j F_i E_i + [2]^2 Q_i E_j F_i E_i \\
&= [2]^2 F_i E_j E_i F_i E_i + [2]^2 E_j F_i E_i \frac{q^2 K_i - q^{-2} K_i^{-1}}{q-q^{-1}} \\
&= [2]^2 F_i E_j F_i E_i^2 + [2]^2 F_i E_i E_j \frac{q^2 K_i - q^{-2} K_i^{-1}}{q-q^{-1}} \\
&\quad + [2]^2 E_j F_i E_i \frac{q^2 K_i - q^{-2} K_i^{-1}}{q-q^{-1}}, \\
\eta_6 &= -\left(1+q^{-2}\right) E_i E_j F_i^2 E_i = -\left(1+q^{-2}\right) E_j E_i F_i^2 E_i \\
&= -\left(1+q^{-2}\right) E_j F_i^2 E_i^2 - [2]\left(1+q^{-2}\right) E_j F_i E_i \bar{x}_i, \\
\eta_7 &= q^2 E_i^2 F_i^2 E_j \\
&= q^2 F_i^2 E_i^2 E_j + [2]^2 q^2 F_i E_i E_j \frac{q^2 K_i - q^{-2} K_i^{-1}}{q-q^{-1}} + [2]^2 q^2 P_i E_j, \\
\eta_8 &= -\left(1+q^2\right) E_i^2 F_i E_j F_i = -\left(1+q^2\right) F_i E_j E_i^2 F_i - [2]\left(1+q^2\right) E_j E_i F_i \bar{x}_i \\
&= -\left(1+q^2\right) F_i E_j F_i E_i^2 - [2]\left(1+q^2\right) F_i E_i E_j \bar{x}_i - [2]\left(1+q^2\right) E_j E_i F_i \bar{x}_i, \\
\eta_9 &= E_i^2 E_j F_i^2 = E_j E_i^2 F_i^2 = E_j F_i^2 E_i^2 + [2]^2 E_j F_i E_i Q_i + [2]^2 E_j P_i.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\eta_1 + \eta_4 + \eta_7 &= -[2]\left(1+q^2\right) F_i E_i E_j \frac{(q^4+q^2)K_i - (q^{-4}+q^{-2})K_i^{-1}}{q^2-q^{-2}} \\
&\quad + [2]^2 q^2 F_i E_i E_j \frac{q^2 K_i - q^{-2} K_i^{-1}}{q-q^{-1}} + [2]^2 q^2 P_i E_j, \qquad (6.7) \\
\eta_2 + \eta_5 + \eta_8 &= [2]^2 F_i E_i E_j \frac{q^2 K_i - q^{-2} K_i^{-1}}{q-q^{-1}} + [2]^2 E_j F_i E_i \frac{q^2 K_i - q^{-2} K_i^{-1}}{q-q^{-1}} \\
&\quad - [2]\left(1+q^2\right) F_i E_i E_j \bar{x}_i - [2]\left(1+q^2\right) E_j E_i F_i \bar{x}_i, \qquad (6.8) \\
\eta_3 + \eta_6 + \eta_9 &= -[2]\left(1+q^2\right) E_j F_i E_i \bar{x}_i + [2]^2 E_j F_i E_i Q_i + [2]^2 E_j P_i. \qquad (6.9)
\end{aligned}
$$

By (6.7), (6.8) and (6.9) the right coefficient of $F_i E_i E_j$ in $\sum_{t=1}^{9} \eta_t$ is

$$
[2]^2 \left\{ -\frac{(q^5+q^3)K_i - (q^{-3}+q^{-1})K_i^{-1}}{q^2-q^{-2}} + \frac{q^4 K_i - K_i^{-1}}{q-q^{-1}} \right.
$$

$$+\frac{q^2K_i-q^{-2}K_i^{-1}}{q-q^{-1}}-\frac{(q+q^3)K_i-(q+q^{-1})K_i^{-1}}{q^2-q^{-2}}\Bigg\}=0,$$

while the right coefficient of $E_jF_iE_i$ in $\sum_{t=1}^{9}\eta_t$ is

$$[2]^2\left\{\frac{q^2K_i-q^{-2}K_i^{-1}}{q-q^{-1}}-q^{-1}\frac{(1+q^2)K_i-(1+q^{-2})K_i^{-1}}{q^2-q^{-2}}+\frac{K_i-K_i^{-1}}{q-q^{-1}}\right\}=0.$$

It follows that

$$\begin{aligned}\sum_{t=1}^{9}\eta_t &= [2]^2qE_jF_iE_i\bar{x}_i-[2]\big(1+q^2\big)E_jE_iF_i\bar{x}_i\\ &\quad+[2]^2q^2P_iE_j+[2]^2E_jP_i\\ &=[2]^2\big\{-qE_jQ_i\bar{x}_i+q^2P_iE_j+E_jP_i\big\}\\ &=[2]^2E_j\Bigg\{-qQ_i\bar{x}_i+q^2\frac{q^4(1+q^{-2})K_{2i}+q^{-4}(1+q^2)K_{2i}^{-1}}{[2]^2(q-q^{-1})^2}\\ &\quad-\frac{q^2}{(q-q^{-1})^2}+\frac{(1+q^{-2})K_{2i}+(1+q^2)K_{2i}^{-1}}{[2]^2(q-q^{-1})^2}-\frac{1}{(q-q^{-1})^2}\Bigg\}\\ &=[2]^2q^2E_jK_{2i},\end{aligned}$$

which means, by (6.6), that $T_iT_i'(E_j)=E_j$. This completes the proof. □

Up to now the proof of Theorem 2.1 is completed.

7. Braid group relations

In this section we prove Theorem 2.2, i.e., the Lusztig symmetries T_i's $(1\leqslant \imath\leqslant \ell+2)$ satisfy braid group relations. Theorem 2.2 follows by following Propositions 7.1, 7.2.

Proposition 7.1. *Assume that $a_{ij}=0$. Then $T_iT_j=T_jT_i$.*

Proof. By definition of the fundamental reflections r_i's of Weyl group of Γ, for any $\beta\in\Gamma$ it follows that $r_ir_j(\beta)=r_jr_i(\beta)$. So we have that

$$T_iT_j(K_\beta)=T_i(K_{r_j(\beta)})=K_{r_ir_j(\beta)}=K_{r_jr_i(\beta)}=T_jT_i(K_\beta).$$

Also,

$$\begin{aligned}T_iT_j\big(D_k^{\pm}\big)&=T_i\big(D_k^{\pm}K_j^{\pm\delta_{j,\ell+k}}\big)=T_i\big(D_k^{\pm}\big)T_i(K_j)^{\pm\delta_{j,k+\ell}}\\ &=D_k^{\pm}K_i^{\pm\delta_{i,k+\ell}}K_j^{\pm\delta_{j,k+\ell}}=T_jT_i\big(D_k^{\pm}\big)\end{aligned}$$

for $k=1,2$. So, by the involution Φ (2.4) of U_q it suffices to check that

$$T_iT_j(E_k)=T_jT_i(E_k)\quad \text{for } 1\leqslant k\leqslant \ell+2. \tag{7.1}$$

In the case either $k=i$ or $k=j$, it is easy to see that (7.1) holds. From now on we assume that $k\neq i$ and $k\neq j$. There are following cases to be checked.

(i) Assume that $a_{ki}=a_{kj}=0$. Then (7.1) is clear.

(ii) Assume that $a_{ki}=0$ and $a_{kj}=-1$. Since $a_{ij}=0$, it follows that

$$\begin{aligned}T_iT_j(E_k)&=-T_i(E_j)T_i(E_k)+q^{-1}T_i(E_k)T_i(E_j)\\&=-E_jE_k+q^{-1}E_kE_j=T_j(E_k)=T_jT_i(E_k),\end{aligned}$$

which means that (7.1) holds in this case.

(iii) Assume that $a_{ki}=-1$ and $a_{kj}=0$. This is similar to (ii). In fact we have that $T_jT_i(E_k)=T_i(E_k)=T_iT_j(E_k)$, which means that (7.1) holds in this case.

(iv) Assume that $a_{ki}=a_{kj}=-1$. Since $a_{ij}=0$, it follows that $E_iE_j=E_jE_i$ and $T_i(E_j)=E_j$ and hence

$$\begin{aligned}T_iT_j(E_k)&=T_i\big(-E_jE_k+q^{-1}E_kE_j\big)=-E_jT_i(E_k)+q^{-1}T_i(E_k)E_j\\&=-E_j\big(-E_iE_k+q^{-1}E_kE_i\big)+q^{-1}\big(-E_iE_k+q^{-1}E_kE_i\big)E_j\\&=E_i\big(E_jE_k-q^{-1}E_kE_j\big)-q^{-1}\big(E_jE_k-q^{-1}E_kE_j\big)E_i\\&=-E_iT_j(E_k)+q^{-1}T_j(E_k)E_i=T_jT_i(E_k),\end{aligned}$$

which means that (7.1) holds in this case.

(v) Assume that $a_{ki}=0$ and $a_{kj}=2$. Note that $k\neq i$ and $k\neq j$. Then

$$\begin{aligned}T_iT_j(E_k)&=\frac{q^2}{[2]}\big\{\big(T_i(E_k)\big)^2T_i(E_k)-(1+q{-}2)T_i(F_j)T_i(E_k)T_i(F_j)\\&\qquad+q^{-2}T_i(E_k)\big(T_i(F_j)\big)^2\big\}\\&=\frac{q^2}{[2]}\big\{F_j^2E_k-\big(1+q^{-2}\big)F_jE_kF_j+q^{-2}E_kF_j^2\big\}\\&=T_j(E_k)=T_jT_i(E_k),\end{aligned}$$

which means that (7.1) holds in this case.

(vi) Assume that $a_{ki}=2$ and $a_{kj}=0$. This is completely similar to (v).

(vii) Assume that $a_{ik}=-1$ and $a_{kj}=2$. Since $E_iF_j=F_jE_i$ and $T_i(F_j)=F_j$, it follows that

$$
\begin{aligned}
T_iT_j(E_k) &= \frac{q^2}{[2]}T_i\bigl\{F_j^2E_k-\bigl(1+q^{-2}\bigr)F_jE_kF_j+q^{-2}E_kF_j^2\bigr\}\\
&= \frac{q^2}{[2]}\bigl\{F_j^2\bigl(-E_iE_k+q^{-1}E_kE_i\bigr)+q^{-2}\bigl(-E_iE_k+q^{-1}E_kE_i\bigr)F_j^2\\
&\quad-\bigl(1+q^{-2}\bigr)F_j\bigl(-E_iE_k+q^{-1}E_kE_i\bigr)F_j\bigr\}\\
&= \frac{q^2}{[2]}\bigl\{-E_i\bigl(F_j^2E_k-\bigl(1+q^{-2}\bigr)F_jE_kF_j+q^{-2}E_kF_j^2\bigr)\\
&\quad+q^{-1}\bigl(F_j^2E_k-\bigl(1+q^{-2}\bigr)F_jE_kF_j+q^{-2}E_kF_j^2\bigr)E_i\\
&= -E_iT_j(E_k)+q^{-1}T_j(E_k)E_i=T_jT_i(E_k),
\end{aligned}
$$

which means that (7.1) holds in this case.

(viii) Assume that $a_{ki}=2$ and $a_{kj}=-1$. This is similar to (vii). This completes the proof. □

Proposition 7.2. *Assume that $a_{ij}=-1$. Then $T_iT_jT_i=T_jT_iT_j$.*

Since $a_{ij}=-1$, it follows that in the Weyl group W of Γ it holds that $r_ir_jr_i=r_jr_ir_j$. Therefore for any $\beta\in\Gamma$ we have that

$$T_iT_jT_i(K_\beta)=K_{r_ir_jr_i(\beta)}=T_jT_iT_j(K_\beta).$$

Moreover, for $k=1,2$, we have that

$$T_iT_jT_i\bigl(D_k^\pm\bigr)=D_k^\pm(K_{i+j})^{\pm\delta_{i,\ell+k}\pm\delta_{j,\ell+k}}=T_jT_iT_j\bigl(D_k^\pm\bigr).$$

So, by the involution Φ (2.4) of U_q it suffices to show that

$$T_iT_jT_i(E_k)=T_jT_iT_j(E_k)\quad\text{for } 1\leqslant k\leqslant \ell+2. \tag{7.2}$$

If $k=i$ then (7.2) becomes that $T_iT_jT_i(E_i)=T_jT_iT_j(E_i)$. Since $a_{ij}=-1$ and hence $K_i^{-1}T_i(F_j)=qT_i(F_j)K_i^{-1}$, $E_iF_j=F_jE_i$, it follows that

$$
\begin{aligned}
T_iT_jT_i(E_i) &= -T_i\bigl(T_j(F_i)T_j(K_i)\bigr)=-T_i\bigl((-F_iF_j+qF_jF_i)K_{i+j}\bigr)\\
&= \bigl(T_i(F_i)T_i(F_j)-qT_i(F_j)T_i(F_i)\bigr)T_i(K_{i+j})\\
&= \bigl(-q^{-1}E_iT_i(F_j)+q^{-1}T_i(F_j)E_i\bigr)K_{-i+j}\\
&= -q^{-1}\{-E_iF_jF_i+qE_iF_iF_j+F_jF_iE_i-qF_iF_jE_i\}K_{-i+j}\\
&= -q^{-1}\{-F_jE_iF_i+F_jF_iE_i+qE_iF_iF_j-qF_iE_iF_j\}K_{-i+j}\\
&= -q^{-1}(-F_jQ_i+qQ_iF_j)K_{-i+j}=-F_jK_j.
\end{aligned}
$$

By a similar way as above we have that $T_jT_iT_j(E_i) = -F_jK_j$. It follows that (7.2) holds in this case. By (iii) of Lemma 4.2 and a similar computation as above we have that $T_iT_jT_i(E_j) = -F_iK_i = T_jT_iT_j(E_j)$, which means that (7.2) holds in the case that $k = j$. So, it remains to check (7.2) in the case that $k \neq i$ and $k \neq j$. We check it in following lemmas. At first we consider the easier cases.

Lemma 7.1. *Assume that $a_{ki} \neq 2$ and $a_{kj} \neq 2$. Then* (7.2) *holds.*

Proof. (i) Assume that $a_{ki} = a_{kj} = 0$. Then $T_iT_jT_i(E_k) = E_k = T_jT_iT_j(E_k)$.

(ii) Assume that $a_{ki} = 0$ and $a_{kj} = -1$. Then $E_iE_k = E_kE_i$ and

$$\begin{aligned}T_iT_jT_i(E_k) &= T_iT_j(E_k) = T_i\bigl(-E_jE_k + q^{-1}E_kE_j\bigr)\\ &= E_iE_jE_k - q^{-1}E_jE_iE_k - q^{-1}E_kE_iE_j + q^{-2}E_kE_jE_i.\end{aligned}$$

On the other hand, by (iii) of Lemma 4.2, we have $T_jT_i(E_j) = E_i$ and hence

$$T_jT_iT_j(E_k) = E_iE_jE_k - q^{-1}E_iE_kE_j - q^{-1}E_jE_kE_i + q^{-2}E_kE_jE_i.$$

It follows that (7.2) holds in this case.

(iii) Assume that $a_{ki} = -1$ and $a_{kj} = 0$. This is similar to (ii).

(iv) Assume that $a_{ki} = a_{kj} = -1$. To check (7.2) we show that $T_iT_jT_i(E_k)$ is symmetric with respect to indices i, j. This can be done as follows. By Lemma 4.2 we have that $T_iT_j(E_i) = E_j$ and $K_jT_j(E_i) = qT_j(E_i)K_j$. It follows that

$$\begin{aligned}T_iT_jT_i(E_k) &= -T_i\bigl(T_j(E_i)\bigr)T_i\bigl(T_j(E_k)\bigr) + q^{-1}T_i\bigl(T_j(E_k)\bigr)T_i\bigl(T_j(E_i)\bigr)\\ &= -E_jT_i\bigl(T_j(E_k)\bigr) + q^{-1}T_i\bigl(T_j(E_k)\bigr)E_j\\ &= -E_jT_i(E_j)T_i(E_k) + q^{-1}E_jT_i(E_k)T_i(E_j)\\ &\quad - q^{-1}T_i(E_j)T_i(E_k)E_j + q^{-2}T_i(E_k)T_i(E_j)E_j.\end{aligned}$$

Note that

$$\begin{aligned}T_i(E_j)T_i(E_k) &= E_iE_jE_iE_k - q^{-1}E_iE_jE_kE_i - q^{-1}E_jE_i^2E_k + q^{-2}E_jE_iE_kE_i,\\ T_i(E_k)T_i(E_j) &= E_iE_kE_iE_j - q^{-1}E_iE_kE_jE_i - q^{-1}E_kE_i^2E_j + q^{-2}E_kE_iE_jE_i.\end{aligned}$$

So, by Serre relation (2.1) we have that $T_iT_jT_i(E_k) = \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4$, where

$$\begin{aligned}\zeta_1 &= E_jE_iE_jE_iE_k - q^{-1}E_jE_iE_jE_kE_i - q^{-1}E_j^2E_i^2E_k + q^{-2}E_j^2E_iE_kE_i\\ &= E_jE_iE_jE_iE_k - q^{-1}E_jE_iE_jE_kE_i\\ &\quad - q^{-1}\bigl(q + q^{-1}\bigr)E_j^{(2)}E_i^2E_k + q^{-2}E_j^2E_i^{(2)}E_k + q^{-2}E_j^2E_kE_i^{(2)}\\ &= \bigl(E_jE_iE_jE_i - E_j^{(2)}E_i^2\bigr)E_k - q^{-1}E_jE_iE_jE_kE_i + q^{-2}E_j^2E_kE_i^{(2)},\end{aligned}$$

$$
\begin{aligned}
\zeta_2 &= -q^{-1}E_jE_iE_kE_iE_j + q^{-2}E_jE_iE_kE_jE_i \\
&\quad + q^{-2}E_jE_kE_i^2E_j - q^{-3}E_jE_kE_iE_jE_i \\
&= -q^{-1}E_jE_iE_kE_iE_j + q^{-2}E_jE_iE_kE_jE_i + q^{-2}\left(q+q^{-1}\right)E_jE_iE_kE_iE_j \\
&\quad - q^{-2}E_jE_i^2E_kE_j - q^{-3}E_jE_kE_iE_jE_i \\
&= q^{-3}E_jE_iE_kE_iE_j + q^{-2}E_jE_iE_kE_jE_i - q^{-2}E_jE_i^2E_kE_j \\
&\quad - q^{-3}E_jE_kE_iE_jE_i, \\
\zeta_3 &= -q^{-1}E_iE_jE_iE_kE_j + q^{-2}E_iE_jE_kE_iE_j \\
&\quad + q^{-2}E_jE_i^2E_kE_j - q^{-3}E_jE_iE_kE_iE_j, \\
\zeta_4 &= q^{-2}E_iE_kE_iE_j^2 - q^{-3}E_iE_kE_jE_iE_j - q^{-3}E_kE_i^2E_j^2 + q^{-4}E_kE_iE_jE_iE_j \\
&= q^{-2}E_i^{(2)}E_kE_j^2 + q^{-2}E_kE_i^{(2)}E_j^2 - q^{-3}E_iE_kE_jE_iE_j \\
&\quad - q^{-3}\left(q+q^{-1}\right)E_kE_i^{(2)}E_j^2 + q^{-4}E_kE_iE_jE_iE_j \\
&= q^{-4}E_k\left(E_iE_jE_iE_j - E_i^{(2)}E_j^2\right) + q^{-2}E_i^{(2)}E_kE_j^2 - q^{-3}E_iE_kE_jE_iE_j.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
T_iT_jT_i(E_k) &= \zeta_1+\zeta_2+\zeta_3+\zeta_4 \\
&= \left(E_jE_iE_jE_i - E_j^{(2)}E_i^2\right)E_k + q^{-4}E_k\left(E_iE_jE_iE_j - E_i^{(2)}E_j^2\right) \\
&\quad - q^{-1}(E_jE_iE_kE_jE_i + E_iE_jE_kE_iE_j) \\
&\quad + q^{-2}(E_jE_iE_kE_jE_i + E_iE_jE_kE_iE_j) \\
&\quad - q^{-3}(E_jE_kE_iE_jE_i + E_iE_kE_jE_iE_j),
\end{aligned}
$$

which is symmetric with respect to indices i, j by noting that, in particular, from Serre relation (2.1) we have that

$$E_jE_iE_jE_i - E_j^{(2)}E_i^2 = E_iE_jE_iE_j - E_i^{(2)}E_j^2 \quad \text{for } a_{ij} = -1.$$

So $T_iT_jT_i(E_k) = T_jT_iT_j(E_k)$. This completes the proof. □

Lemma 7.2. *The equality* (7.2) *holds in each case of the following.*

(i) $a_{ki} = 0, a_{kj} = 2, a_{ij} = -1$.
(ii) $a_{ki} = 2, a_{kj} = -1, a_{ij} = -1$.

Proof. We show (i) since (ii) is completely similar. At first, by definition of T_i (cf. (2.6), (2.7)) we have that

$$
\begin{aligned}
&[2]q^{-2}T_iT_jT_i(E_k)\\
&\quad=[2]q^{-2}T_iT_j(E_k)\\
&\quad=\big(T_i(F_j)\big)^2E_k-\big(1+q^{-2}\big)T_i(F_j)E_kT_i(F_j)+q^{-2}E_k\big(T_i(F_j)\big)^2
\end{aligned}
$$

and

$$
\begin{aligned}
&[2]q^{-2}T_jT_iT_j(E_k)\\
&\quad=T_jT_i\big\{F_j^2E_k-\big(1+q^{-2}\big)F_jE_kF_j+q^{-2}E_kF_j^2\big\}\\
&\quad=\big(T_j\big(T_i(F_j)\big)\big)^2T_j(E_k)-\big(1+q^{-2}\big)T_j\big(T_i(F_j)\big)T_j(E_k)T_j\big(T_i(F_j)\big)\\
&\qquad+q^{-2}T_j(E_k)\big(T_j\big(T_i(F_j)\big)\big)^2\\
&\quad=F_i^2T_j(E_k)-\big(1+q^{-2}\big)F_iT_j(E_k)F_i+q^{-2}T_j(E_k)F_i^2,
\end{aligned}
$$

where in the last equality we use the fact that $T_j(T_i(F_j))=F_i$ for $a_{ij}=-1$. So it remains to show that

$$
\begin{aligned}
&\big(T_i(F_j)\big)^2E_k-\big(1+q^{-2}\big)T_i(F_j)E_kT_i(F_j)+q^{-2}E_k\big(T_i(F_j)\big)^2\\
&\quad=F_i^2T_j(E_k)-\big(1+q^{-2}\big)F_iT_j(E_k)F_i+q^{-2}T_j(E_k)F_i^2.
\end{aligned}
\tag{7.3}
$$

At first, similar to (5.19), since $a_{ij}=-1$, we have that

$$
\big(T_i(F_j)\big)^2=q^2F_i^2F_j^{(2)}-qF_iF_j^2F_i+F_j^{(2)}F_i^2.
\tag{7.4}
$$

Since $F_iE_k=E_kF_i$, by (7.4) it follows that

$$
\begin{aligned}
\big(T_i(F_j)\big)^2E_k&=q^2F_i^2F_j^{(2)}E_k-qF_iF_j^2E_kF_i+F_j^{(2)}E_kF_i^2,\\
q^{-2}E_k\big(T_i(F_j)\big)^2&=-F_i^2E_kF_j^{(2)}-q^{-1}F_iE_kF_j^2F_i+q^{-2}E_kF_j^{(2)}F_i^2.
\end{aligned}
$$

Also, $T_i(F_j)E_kT_i(F_j)=\kappa_1+\kappa_2+\kappa_3+\kappa_4$, where

$$
\begin{aligned}
\kappa_1&=F_jF_iE_kF_jF_i=F_jE_kF_iF_jF_i=F_jE_kF_i^{(2)}F_j+F_jE_kF_jF_i^{(2)}\\
&=F_iF_jE_kF_iF_j-F_i^{(2)}F_jE_kF_j+F_jE_kF_jF_i^{(2)},\\
\kappa_2&=-qF_jF_iE_kF_iF_j=-qF_jF_i^2E_kF_j\\
&=-\big(1+q^2\big)F_iF_jE_kF_iF_j+\big(1+q^{-2}\big)F_i^{(2)}F_jE_kF_j,\\
\kappa_3&=-qF_iF_jE_kF_jF_i,\\
\kappa_4&=q^2F_iF_jE_kF_iF_j.
\end{aligned}
$$

So we have that

$$-\left(1+q^{-2}\right)T_i(F_j)E_kT_i(F_j)$$
$$=-qF_i^2F_jE_kF_j+\left(q+q^{-1}\right)F_iF_jE_kF_jF_i-q^{-1}F_jE_kF_jF_i^2.$$

It follows that

$$\begin{aligned}
\text{L.H.S. of (7.3)} &= F_i^2\left[q^2F_j^{(2)}E_k-qF_jE_kF_j+E_kF_j^{(2)}\right]\\
&\quad -F_i\left[qF_j^2E_k-\left(q+q^{-1}\right)F_jE_kF_j+q^{-1}E_kF_j^2\right]F_i\\
&\quad +\left[F_j^{(2)}E_k-q^{-1}F_jE_kF_j+q^{-2}E_kF_j^{(2)}\right]F_i^2\\
&= F_i^2\left[q^2F_j^{(2)}E_k-qF_jE_kF_j+E_kF_j^{(2)}\right]\\
&\quad -\left(1+q^{-2}\right)F_i\left[q^2F_j^{(2)}E_k-qF_jE_kF_j+E_kF_j^{(2)}\right]F_i\\
&\quad +q^{-2}\left[q^2F_j^{(2)}E_k-qF_jE_kF_j+E_kF_j^{(2)}\right]F_i^2\\
&= F_i^2T_j(E_k)-\left(1+q^{-2}\right)F_iT_j(E_k)F_i+q^{-2}T_j(E_k)F_i^2\\
&= \text{R.H.S. of (7.3)}
\end{aligned}$$

as required. This completes the proof. □

Lemma 7.3. *The equality* (7.2) *holds in the following cases.*

(i) $a_{ki}=-1$, $a_{kj}=2$ *and* $a_{ij}=2$.
(ii) $a_{ki}=2$, $a_{kj}=-1$ *and* $a_{ij}=2$.

Proof. We prove (i) only since (ii) is similar. Since $a_{ij}=-1$ we have that $T_i(T_j(E_i))=E_j$ and $T_jT_i(F_j))=F_i$ and hence

$$T_iT_jT_i(E_k)=-E_jT_i\left(T_j(E_k)\right)+q^{-1}T_i\left(T_j(E_k)\right)E_j,$$
$$T_jT_iT_j(E_k)=\frac{q^2}{[2]}\left\{F_i^2T_j\left(T_i(E_k)\right)-\left(1+q^{-2}\right)F_iT_j\left(T_i(E_k)\right)F_i+q^{-2}T_j\left(T_i(E_k)\right)F_i^2\right\}.$$

So, it remains to show that

$$\begin{aligned}
&[2]q^{-2}\left\{-E_jT_i\left(T_j(E_k)\right)+q^{-1}T_i\left(T_j(E_k)\right)E_j\right\}\\
&\quad =F_i^2T_j\left(T_i(E_k)\right)-\left(1+q^{-2}\right)F_iT_j\left(T_i(E_k)\right)F_i+q^{-2}T_j\left(T_i(E_k)\right)F_i^2. \qquad (7.5)
\end{aligned}$$

Set $\varphi_{jk}=F_jE_k-q^{-2}E_kF_j$. Since $a_{kj}=2$, by (iii) of Lemma 5.1 we have that $E_jT_j(E_k)=T_j(E_k)E_j+q^2\varphi_{jk}K_j$. By this and the fact that $E_iF_j=F_jE_i$ with a direct computation it follows that

$$T_j\left(T_i(E_k)\right)=-E_j\vartheta+q^{-1}\vartheta E_j+(\varphi_{jk}E_i-qE_i\varphi_{jk})K_j, \qquad (7.6)$$

where

$$\vartheta = q^2\big\{F_j^{(2)}T_i(E_k) - q^{-1}F_jT_i(E_k)F_j + q^{-2}T_i(E_k)F_j^{(2)}\big\}. \tag{7.7}$$

Since $F_jE_i = E_iF_j$, by (7.6) and (7.7) it follows that

$$\begin{aligned}\text{R.H.S. of (7.5)} = {} & -E_j\big(F_i^2\vartheta - \big(1+q^{-2}\big)F_i\vartheta F_i + q^{-2}\vartheta F_i^2\big)\\ & + q^{-1}\big(F_i^2\vartheta - \big(1+q^{-2}\big)F_i\vartheta F_i + q^{-2}\vartheta F_i^2\big)E_j\\ & + \big\{\big[F_i^2\varphi_{jk}E_i - [2]F_i\varphi_{jk}E_iF_i + \varphi_{jk}E_iF_i^2\big]\\ & - q\big[F_i^2E_i\varphi_{jk} - [2]F_iE_i\varphi_{jk}F_i + E_i\varphi_{jk}F_i^2\big]\big\}K_j.\end{aligned}$$

We compute the two parts of the right-hand side of (7.5) as follows. At first we compute the latter part in the above formula. By Lemma 3.3 it follows that

$$\begin{aligned}F_i^2\varphi_{jk}E_i &= F_i^2F_jE_kE_i - q^{-2}E_kF_i^2F_jE_i,\\ -[2]F_i\varphi_{jk}E_iF_i &= -[2]\big(F_iF_jF_iE_kE_i + F_iF_jE_kQ_i\\ &\quad - q^{-2}E_kF_iF_jF_iE_i - q^{-2}F_iE_kF_jQ_i\big),\\ \varphi_{jk}E_iF_i^2 &= F_jF_i^2E_kE_i + [2]F_jF_iE_kx_i\\ &\quad - q^{-2}E_kF_jF_i^2E_i - [2]q^{-2}E_kF_jF_ix_i.\end{aligned}$$

Thus, by Serre relation (2.1) we have that

$$\begin{aligned}&\big[F_i^2\varphi_{jk}E_i - [2]F_i\varphi_{jk}E_iF_i + \varphi_{jk}E_iF_i^2\big]\\ &\quad = -[2]\big\{F_iF_jE_kQ_i - F_jF_iE_kx_i - q^{-2}E_kF_iF_jQ_i + q^{-2}E_kF_jF_ix_i\big\}.\end{aligned}$$

By a completely similar computation we have that

$$\begin{aligned}&q\big[F_i^2E_i\varphi_{jk} - [2]F_iE_i\varphi_{jk}E_i + E_i\varphi_{jk}F_i^2\big]\\ &\quad = [2]\bigg\{q^{-2}E_kF_iF_jx_i - F_iF_jE_k\frac{q^{-1}K_i - qK_i^{-1}}{q-q^{-1}}\\ &\qquad - q^{-2}E_kF_jF_i\frac{q^{-2}K_i - q^2K_i^{-1}}{q-q^{-1}} + F_jF_iE_k\frac{(1+q^{-2})q^{-1}K_i - (1+q^2)qK_i^{-1}}{q^2-q^{-2}}\bigg\}.\end{aligned}$$

It follows that

$$\begin{aligned}&\big[F_i^2\varphi_{jk}E_i - [2]F_i\varphi_{jk}E_iF_i + \varphi_{jk}E_iF_i^2\big]\\ &\qquad - q\big[F_i^2E_i\varphi_{jk} - [2]F_iE_i\varphi_{jk}F_i + E_i\varphi_{jk}F_i^2\big]\\ &\quad = [2]q\big\{T_j(F_i)E_k - q^{-2}E_kT_j(F_i)\big\}K_i^{-1}.\end{aligned}$$

Now we compute the first part of the right-hand side of (7.5) as follows. Since $a_{ki} = -1$, by (iii) of Lemma 4.2 we have that $F_i T_i(E_k) = T_i(E_k)F_i - E_k K_i^{-1}$ and hence $F_i^2 T_i(E_k) = T_i(E_k)F_i^2 - (1+q^2)E_k F_i K_i^{-1}$. It follows that

$$\begin{aligned}
F_i^2 \vartheta &= q^2 F_i^2 F_j^{(2)} T_i(E_k) - q F_i^2 F_j T_i(E_k) F_j + F_i^2 T_i(E_k) F_j^{(2)} \\
&= q^2 F_i^2 F_j^{(2)} T_i(E_k) - q F_i^2 F_j T_i(E_k) F_j \\
&\quad + T_i(E_k) F_i^2 F_j^{(2)} - \bigl(1+q^2\bigr) q^{-2} E_k F_i F_j^{(2)} K_i^{-1}, \\
-\bigl(1+q^{-2}\bigr) F_i \vartheta F_i &= -\bigl(1+q^2\bigr) F_i \bigl\{ F_j^{(2)} T_i(E_k) - q^{-1} F_j T_i(E_k) F_j + q^{-2} T_i(E_k) F_j^{(2)} \bigr\} F_i \\
&= -q F_i F_j^2 F_i T_i(E_k) - \bigl(1+q^2\bigr) F_i F_j^{(2)} E_k K_i^{-1} \\
&\quad + \bigl(1+q^2\bigr) q^{-1} F_i F_j T_i(E_k) F_j F_i - q^{-1} T_i(E_k) F_i F_j^2 F_i \\
&\quad + \bigl(1+q^{-2}\bigr) E_k F_j^{(2)} F_i K_i^{-1}, \\
q^{-2} \vartheta F_i^2 &= \bigl\{ F_j^{(2)} T_i(E_k) - q^{-1} F_j T_i(E_k) F_j + q^{-2} T_i(E_k) F_j^{(2)} \bigr\} F_i^2 \\
&= F_j^{(2)} F_i^2 T_i(E_k) + \bigl(1+q^2\bigr) F_j^{(2)} F_i E_k K_i^{-1} \\
&\quad - q^{-1} F_j T_i(E_k) F_j F_i^2 + q^{-2} T_i(E_k) F_j^{(2)} F_i^2.
\end{aligned}$$

Note that $(T_i(F_j))^2 = q^2 F_i^2 F_j^{(2)} - q F_i F_j^2 F_i + F_j^{(2)} F_i^2$. Summing the above three formulae we have that

$$\begin{aligned}
&F_i^2 \vartheta - \bigl(1+q^{-2}\bigr) F_i \vartheta F_i + q^{-2} \vartheta F_i^2 \\
&\quad = \bigl(T_i(F_j)\bigr)^2 T_i(E_k) + q^{-2} T_i(E_k) \bigl(T_i(F_j)\bigr)^2 \\
&\qquad - q F_i^2 F_j T_i(E_k) F_j + [2] F_i F_j T_i(E_k) F_j F_i - q^{-1} F_j T_i(E_k) F_j F_i^2 \\
&\qquad + \bigl\{ -\bigl(1+q^{-2}\bigr) E_k F_i F_j^{(2)} - \bigl(1+q^2\bigr) F_i F_j^{(2)} E_k \\
&\qquad + \bigl(1+q^{-2}\bigr) E_k F_j^{(2)} F_i + \bigl(1+q^2\bigr) F_j^{(2)} F_i E_k \bigr\} K_i^{-1}.
\end{aligned}$$

By using Serre relation (2.1) and a similar computation as above it follows that

$$\begin{aligned}
-q F_i^2 F_j T_i(E_k) F_j &= -\bigl(1+q^2\bigr) F_i F_j T_i(E_k) F_i F_j + q F_j F_i T_i(E_k) F_i F_j \\
&\quad + [2] F_i F_j E_k F_j K_i^{-1} - F_j F_i E_k F_j K_i^{-1}, \\
-q^{-1} F_j T_i(E_k) F_j F_i^2 &= -\bigl(1+q^{-2}\bigr) F_j F_i T_i(E_k) F_j F_i + q^{-1} F_j F_i T_i(E_k) F_i F_j \\
&\quad - [2] F_j E_k F_j F_i K_i^{-1} + F_j E_k F_i F_j K_i^{-1}.
\end{aligned}$$

Since

$$\begin{aligned}
&T_i(F_j)T_i(E_k)T_i(F_j)\\
&\quad = F_jF_iT_i(E_k)F_jF_i - qF_jF_iT_i(E_k)F_iF_j - qF_iF_jT_i(E_k)F_jF_i\\
&\qquad + q^2F_iF_jT_i(E_k)F_iF_j,
\end{aligned}$$

it follows that

$$\begin{aligned}
&-qF_i^2F_jT_i(E_k)F_j - q^{-1}F_jT_i(E_k)F_jF_i^2 + [2]F_iF_jT_i(E_k)F_jF_i\\
&\quad = -\left(1+q^{-2}\right)T_i(F_j)T_i(E_k)T_i(F_j)\\
&\qquad + \left\{[2]F_iF_jE_kF_j + F_jE_kF_iF_j - F_jF_iE_kF_j - [2]F_jE_kF_jF_i\right\}K_i^{-1},
\end{aligned}$$

and hence

$$\begin{aligned}
&F_i^2\vartheta - \left(1+q^{-2}\right)F_i\vartheta F_i + q^{-2}\vartheta F_i^2\\
&\quad = \left(T_i(F_j)\right)^2T_i(E_k) - \left(1+q^{-2}\right)T_i(F_j)T_i(E_k)T_i(F_j) + q^{-2}T_i(E_k)\left(T_i(F_j)\right)^2\\
&\qquad + \varrho K_i^{-1},
\end{aligned}$$

where

$$\begin{aligned}
\varrho = {} & -q^{-1}E_kF_iF_j^2 - qF_iF_j^2E_k + q^{-1}E_kF_j^2F_i + qF_j^2F_iE_k\\
& + [2]F_iF_jE_kF_j - [2]F_jE_kF_jF_i. \qquad (7.8)
\end{aligned}$$

So we have that

$$\begin{aligned}
\text{R.H.S. of (7.5)} = {} & -E_j\left\{\left(T_i(F_j)\right)^2T_i(E_k) + q^{-2}T_i(E_k)\left(T_i(F_j)\right)^2\right.\\
& \left. - \left(1+q^{-2}\right)T_i(F_j)T_i(E_k)T_i(F_j) + \varrho K_i^{-1}\right\}\\
& + q^{-1}\left\{\left(T_i(F_j)\right)^2T_i(E_k) + q^{-2}T_i(E_k)\left(T_i(F_j)\right)^2\right.\\
& \left. - \left(1+q^{-2}\right)T_i(F_j)T_i(E_k)T_i(F_j) + \varrho K_i^{-1}\right\}E_j\\
& + [2]q\left\{T_j(F_i)E_k - q^{-2}E_kT_j(F_i)\right\}K_i^{-1}K_j. \qquad (7.9)
\end{aligned}$$

Note that

$$\begin{aligned}
T_i\left(T_j(E_k)\right) &= \frac{q^2}{[2]}T_i\left\{F_j^2E_k - \left(1+q^{-2}\right)F_jE_kF_j + q^{-2}E_kF_j^2\right\}\\
&= \frac{q^2}{[2]}\left\{\left(T_i(F_j)\right)^2T_i(E_k) + q^{-2}T_i(E_k)\left(T_i(F_j)\right)^2\right.\\
&\qquad \left. - \left(1+q^{-2}\right)T_i(F_j)T_i(E_k)T_i(F_j)\right\},
\end{aligned}$$

which means that

$$\begin{aligned}\text{L.H.S. of (7.5)} = &-E_j\big\{\big(T_i(F_j)\big)^2T_i(E_k)+q^{-2}T_i(E_k)\big(T_i(F_j)\big)^2\\&-\big(1+q^{-2}\big)T_i(F_j)T_i(E_k)T_i(F_j)\big\}\\&+q^{-1}\big\{\big(T_i(F_j)\big)^2T_i(E_k)+q^{-2}T_i(E_k)\big(T_i(F_j)\big)^2\\&-\big(1+q^{-2}\big)T_i(F_j)T_i(E_k)T_i(F_j)\big\}E_j.\end{aligned}\tag{7.10}$$

By (7.9) and (7.10), to prove (7.5) it suffices to show that

$$-E_j\varrho+\varrho E_j+[2]q\big\{T_j(F_i)E_k-q^{-2}E_kT_j(F_i)\big\}K_j=0,\tag{7.11}$$

by noting that $K_i^{-1}E_j=qE_jK_i^{-1}$, where ϱ is given by (7.8).

Let us compute $E_j\varrho$ by using Lemma 3.3 and the facts that $E_jF_i=F_jE_i$, $E_iE_k=E_kE_i$ as follows.

$$\begin{aligned}E_j\big(-q^{-1}E_kF_iF_j^2\big)&=-q^{-1}E_kF_iF_j^2E_j-[2]q^{-1}E_kF_iF_jx_j,\\E_j\big(-qF_iF_j^2E_k\big)&=-qF_iF_j^2E_kE_j\\&\quad-[2]qF_iF_jE_k\frac{(1+q^2)K_j-(1+q^{-2})K_j^{-1}}{q^2-q^{-2}},\\E_j\big(q^{-1}E_kF_j^2F_i\big)&=q^{-1}E_kF_j^2F_iE_j+[2]q^{-1}E_kF_jF_iQ_j,\\E_j\big(qF_j^2F_iE_k\big)&=qF_j^2F_iE_kE_j\\&\quad+[2]qF_jF_iE_k\frac{(1+q^{-2})q^3K_j-(1+q^2)q^{-3}K_j^{-1}}{q^2-q^{-2}},\\E_j\big([2]F_iF_jE_kF_j\big)&=[2]F_iF_jE_kF_jE_j+[2]F_iF_jE_kQ_j+[2]E_kF_iF_jQ_j,\\E_j\big(-[2]F_jE_kF_jF_i\big)&=-[2]F_jE_kF_jF_iE_j-[2]F_jF_iE_k\frac{qK_j-q^{-1}K_j^{-1}}{q-q^{-1}}\\&\quad-[2]E_kF_jF_i\frac{qK_j-q^{-1}K_j^{-1}}{q-q^{-1}}.\end{aligned}$$

So we have that

$$\begin{aligned}&-E_j\varrho+\varrho E_j\\&\quad=E_kF_iF_j\big\{[2]q^{-1}x_j-[2]Q_j\big\}-E_kF_jF_i\left\{[2]q^{-1}Q_j-[2]\frac{qK_j-q^{-1}K_j^{-1}}{q-q^{-1}}\right\}\\&\qquad+F_iF_jE_k\left\{[2]q\frac{(1+q^2)K_j-(1+q^{-2})K_j^{-1}}{q^2-q^{-2}}-[2]Q_j\right\}\end{aligned}$$

$$
\begin{aligned}
&- F_j F_i E_k \left\{[2]q \frac{(1+q^{-2})q^3 K_j - (1+q^2)q^{-3} K_j^{-1}}{q^2 - q^{-2}} - [2]\frac{qK_j - q^{-1}K_j^{-1}}{q - q^{-1}}\right\} \\
&= [2]\left\{q^{-1}E_k[-F_iF_j + qF_jF_i] - q[-F_iF_j + qF_jF_i]E_k\right\}K_j \\
&= -[2]q\left\{T_j(F_i)E_k - q^{-2}E_kT_j(F_i)\right\}K_j,
\end{aligned}
$$

which means (7.11) holds. This completes the proof. □

Up to now the proof of Theorem 2.2 is completed.

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